Standards for Mathematical Practice: Commentary and Elaborations for 6–8

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For discussion of the Elaborations and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Standards for Mathematical Practice, annotated for the 6–8 classroom

The Common Core State Standards describe the Standards for Mathematical Practice this way:

> The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

In this document we provide two different ways of adapting the language of the practice standards to the 6–8 setting.

In this section we provide annotated versions of the standards that provide additional interpretation of the standards appropriate for 6–8 classrooms. This section is intended for people who want to understand how the original language of the standards applies in 6–8.

In the next section we provide *elaborations* of the standards: narrative descriptions that integrate the annotations from the first section and provide a coherent description of how the practice standards play out in the 6–8 classroom.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.• They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.• Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

• For example, to understand why a 20% discount followed by a 20% markup does not return an item to its original price, a middle school student might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item priced at $1.00 or $10.00.

• For example, middle school students should navigate among tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations.• They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.• Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.•

• For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems.

• For example, they can solve problems involving unit rates by representing the situations in equation form.

• For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures.• They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.• They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.• Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and – if there is a flaw in an argument – explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.• Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

• For example, students might conjecture that the diagonals of a parallelogram bisect each other, after having experimented with a representative selection of possible parallelograms.

• An important use of counterexamples in Grades 6–8 is the use of numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$.

• For example they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies.

• Proficient middle school students progress from arguing exclusively through concrete referents such as physical objects and pictorial referents, to also including symbolic representations such as expressions and equations.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.• They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.•

• For example, they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability.

• For example, they can recognize the limitations of linear models in certain situations, such as representing the amounts of stretch in a bungee cord for people of different weights.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.• For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.• Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.•

• In middle school, students might use graphs to model functions, algebra tiles to see how properties of operations apply to algebraic expressions, graphing calculators to solve systems of equations, and dynamic geometry software to discover properties of parallelograms.

• For example, they might use a spreadsheet simulation to answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?

• A middle school student might use a computer applet demonstrating Archimedes' procedure for approximating the value of π.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning.• When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient elementary students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.• In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

• For example, they can use the definition of rational numbers to explain why a number is irrational, and describe congruence and similarity in terms of transformations in the plane.

• In middle school they accurately apply scientific notation to large numbers and use measures of center to describe data sets.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have.• Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.• They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.•

• In middle school, students might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation $3x = 2y$ represents a proportional relationship with a unit rate of $3/2 = 1.5$.

• They might recognize how the Pythagorean theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism.

• Proficient middle school students can also step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by three. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.• By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1,2)$ with slope 3, middle school students might abstract the equation $(y-2)/(x-1)=3$.• Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details.• They continually evaluate the reasonableness of their intermediate results.

• By working with tables of equivalent ratios, middle school students can deduce the corresponding multiplicative relationships and connections to unit rates.

• In middle school geometry, noticing the regularity with which interior angle sums increase with the number of sides in a polygon leads to a general formula for the interior angle sum of an n-gon.

• Middle school students learn to see subtraction as addition of the opposite, and use this is a general purpose tool for collecting like terms in linear expressions.

6–8 elaborations of the standards for mathematical practice

The following elaborations of the practice standards integrate the commentary from the previous section into a single narrative describing a 6–8 version of each standard for mathematical practice. This way of looking at the standards might be more useful than the previous section in working with 6–8 teachers.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students set out to understand a problem and then look for entry points to its solution. They analyze problem conditions and goals, translating, for example, verbal descriptions into mathematical expressions, equations, or drawings as part of the process. They consider analogous problems, and try special cases and simpler forms of the original in order to gain insight into its solution. To understand why a 20% discount followed by a 20% markup does not return an item to its original price, they might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item priced at $1.00 or $10.00. While working on a problem, they monitor and evaluate their progress and change course if necessary. Mathematically proficient students can explain how alternate representations of problem conditions relate to each other. For example, they can navigate among tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change. Mathematically proficient students check their answers to problems and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and compare different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and relationships in problem situations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. They represent problem situations using symbols and then manipulate those symbols in search of a solution. They can, for example, solve problems involving unit rates by representing the situations in equation form. Mathematically proficient students also pause as needed during the manipulation process to double-check or apply referents for the symbols involved. In the process, they can look back at symbol referents and the applicable units of measure to clarify or inform solution steps. Quantitative reasoning also entails knowing and flexibly using different properties of operations and objects. For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use assumptions, definitions, and previously established results in constructing arguments. They make and explore the validity of conjectures. For example, students might conjecture that the diagonals of a parallelogram bisect each other, after having experimented with a representative selection of possible parallelograms. They can recognize and appreciate the use of counterexamples, for example, using numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$. Mathematically proficient students justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and – if there is a flaw in an argument – explain what it is. They can construct formal arguments, progressing from the use of concrete referents such as objects and actions and pictorial referents such as drawings and diagrams to symbolic representations such as expressions and equations. They can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This might be as simple as translating a verbal description to a mathematical expression. It might also entail applying proportional reasoning to plan a school event or using a set of linear inequalities to analyze a problem in the community. Mathematically proficient students are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. For example, they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability. They are able to identify important quantities in a given relationship such as rates of change and represent situations using such tools as diagrams, tables, graphs, flowcharts and formulas. They can analyze their representations mathematically, use the results in the context of the situation, and then reflect on whether the results make sense, possibly improving the model if it has not served its purpose. For example, they can recognize the limitations of linear models in certain situations, such as representing the amounts of stretch in a bungee cord for people of different weights.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a graphing calculator, a spreadsheet, a statistical package, or dynamic geometry software. Proficient students make sound decisions about when each of these tools might be helpful, recognizing both the insights to be gained and their limitations. For example, they use graphs to model functions, algebra tiles to see how properties of operations apply to algebraic expressions, graphing calculators to solve systems of equations, and dynamic geometry software to discover properties of parallelograms. When making mathematical models, they know that technology can enable them to visualize the results of their assumptions, to explore consequences, and to compare predictions with data. For example, they might use a spreadsheet simulation to answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood? Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts, such as a computer applet demonstrating Archimedes' procedures for approximating the value of π.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. For example, they can use the definition of rational numbers to explain why a number is irrational and describe congruence and similarity in terms of transformations in the plane. They state the meaning of the symbols they choose, consistently and appropriately, such as inputs and outputs represented by function notation. They are careful about specifying units of measure, distinguishing, for example, between linear and area measures. They label axes to display the correct correspondence between quantities in a problem, such as the intervals and frequencies on the axes of a histogram. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context, and make explicit use of definitions. For example, they accurately apply scientific notation to large numbers and use measures of center to describe data sets.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. They might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation $3x = 2y$ represents a proportional relationship with a unit rate of $3/2 = 1.5$. They might recognize how the Pythagorean theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism. They also can step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by six. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Working with tables of equivalent ratios, they might deduce the corresponding multiplicative relationships and connections to unit rates. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity with which interior angle sums increase with the number of sides in a polygon might lead them to the general formula for the interior angle sum of an n-gon. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. For example, they learn to see subtraction as addition of the opposite, and use this is a general purpose tool for collecting like terms in linear expressions. They continually evaluate the reasonableness of their intermediate results.