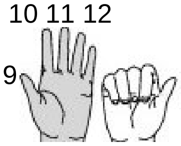
Appendix F: Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away.

Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Adding (8 + 6 = □): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown.

Subtracting (14 – 8 = □): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown added. Use this strategy for Take From/Result Unknown.

Levels	**8 + 6 = 14**	**14 - 8 = 6**
Level 1: Count all	Count All a b 1 2 3 4 5 6 7 8 1 2 3 4 5 6 o o o o o o o o o o o o o o 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~o o o o o o o o~~ o o o o o o 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b
Level 2: Count on	Count On 8 o o o o o o o o o o o o o o 9 10 11 12 13 14 8	To solve 14 – 8 I count on 8 + ? = 14 10 11 12 I took away 8 9  14 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten o o o o o o o o o o o o o o 10 + 4 o o o o o o o o o o o o o o 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~o o o o o o o o~~ o o o o o o 8 + 2 + 4 6 8 + 6 = 14
Doubles = *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2. Counting On.

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imagined objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on can be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words and extended experience with Level 1 methods in kindergarten.

> Adding (e.g., 8 + 6 = □) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
>
> Finding an unknown addend (e.g., 8 + □ = 14): One counts on from the known addend. The keeping track method is monitored so that counting on stops when the known total has been reached. The keeping track method tells the unknown addend.
>
> Subtracting (14 – 8 = □): One thinks of subtracting as finding the unknown addend, as 8 + □ = 14 and uses counting on to find an unknown addend (as above).

The problems in Table 1 in the Glossary which can be solved by Level 1 methods in kindergarten can also be solved using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting). Level 2 and 3 methods are generally used in grades one and two.

> Finding an unknown addend (e.g., 8 + □ = 14) is used for Add To/Change Unknown, Put Together/Take Apart/Addend Unknown, and Compare/Difference Unknown. It is also used for Take From/Change Unknown (14 - □ = 8) after a student has decomposed the total into two addends, which means they can represent the situation as 14 – 8 = □.
>
> Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in Table 1 of the Glossary). Grade one students do not necessarily master the Compare Bigger Unknown or

Smaller Unknown problems with the misleading language in the bottom row of Table 1 of the Glossary.

Solving an equation such as 6 + 8 = □ by counting on from 8 relies on the understanding that 8 + 6 gives the same total, an implicit use of the commutative property without the accompanying written representation 6 + 8 = 8 + 6.

Level 3. Convert to an Easier Equivalent Problem.

Decompose an addend and compose a part with another addend.
These methods can be used to add or to find an unknown addend (and thus to subtract). These methods implicitly use the associative property.

Adding

Make a ten. E.g., for 8 + 6 = □,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so 8 + 6 becomes 10 + 4.
Doubles plus or minus 1. E.g., for 6 + 7 = □,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so 6 + 7 becomes 12 + 1.

Finding an unknown addend.

Make a ten. E.g., for 8 + □ = 14,

$$8 + \underline{2} = 10 \text{ and } \underline{4} \text{ more makes } 14.\ \underline{2 + 4} = 6.$$

So 8 + □ = 14 is done as two steps: how many up to ten and how many over ten (which can be seen in the ones place of 14).
Doubles plus or minus 1. E.g., for 6 + □ = 13,

$$6 + \underline{6 + 1} = 12 + 1.\ \underline{6 + 1} = 7.$$

So 6 + □ = 13 is done as two steps: how many up to 12 (6 + 6) and how many from 12 to 13.

Subtracting.

Thinking of subtracting as finding an unknown addend. E.g., solve 14 – 8 = □ or 13 – 6 = □ as 8 + □ = 14 or 6 + □ = 13 by the above methods (make a ten or doubles plus or minus 1).
Make a ten by going down over ten. E.g., 14 – 8 = □ can be done in two steps by going down over ten: 14 – 4 (to get to 10) – 4 = 6.

The Level 1 and Level 2 problem types can be solved using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend as shown above by methods at any level, but usually at Level 2 or 3. Many students only show in their writing part of this multi-step process of re-representing the situation.

> Students re-represent Add To/Start Unknown □ + 6 = 14 situations as 6 + □ = 14 by using the commutative property (formally or informally).
>
> Students re-represent Take From/Start Unknown □ - 8 = 6 situations by reversing as 6 + 8 = □, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare misleading language situations can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct situation by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same diagrams used for the other versions of compare situations; focusing on which quantity is bigger and which is smaller helps to overcome the misleading language.

Some students may solve Level 3 problem types by doing the above re-representing but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, facilitating such re-representing. Labels on diagrams can help connect the parts of the diagram to the corresponding parts of the situation. But students may know and understand things that they may not use for a given solution of a problem as they increasingly do various representing and re-representing steps mentally.

(Adapted from Progressions, K-5 CC and OA 2011)