Precalculus

Introduction

Precalculus combines the trigonometric, geometric, and algebraic concepts needed to prepare students for the study of Calculus, and strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students intending to study calculus, physics, and other sciences, and/or engineering in college. The main topics in the course are complex numbers, rational functions, trigonometric functions and their inverses, inverse functions, vectors and matrices, and parametric and polar curves. Because the standards for this course are mostly (+) standards, students selecting this Precalculus course should have met the college and career ready standards of the previous courses in the Integrated or Traditional Pathways. This course is highly suggested as preparation before taking a standard Calculus course that would lead to taking an Advanced Placement Calculus exam.

What Students Learn in Precalculus

Overview

In Precalculus, students extend their work with complex numbers begun in Mathematics III or Algebra II to see that the complex numbers can be represented in the Cartesian plane and that operations with complex numbers have a geometric interpretation. They connect their understanding of trigonometry and the geometry of the plane to express complex numbers in polar form.

Students begin working with vectors, representing them geometrically and performing operations with them. They connect the notion of vectors to the complex numbers. Students also work with matrices and their operations, experiencing for the first time an algebraic system in which multiplication is not commutative. Finally, they see the connection between matrices and transformations of the plane, namely, that a vector in the plane can be multiplied by a 2×2 matrix to produce another vector, and they work with matrices from the point of view of transformations. They also find inverse matrices and use matrices to represent and solve linear systems.

Students extend their work with trigonometric functions, investigating the reciprocal functions secant, cosecant, and cotangent and their graphs and properties. They find inverse trigonometric functions by appropriately restricting the domains of the standard trigonometric functions and use them to solve problems that arise in modeling contexts.

While students have worked previously with parabolas and circles, they now work with ellipses and hyperbolas. They also work with polar coordinates and curves defined parametrically, and connect these to their other work with trigonometry and complex numbers.

Finally, students work with more complicated rational functions, graphing them and determining zeros, *y*-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points.

Connecting Standards for Mathematical Practice and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics in* the classroom and, to the extent possible, content instruction should include attention to appropriate practice standards. The table below gives examples of how students can engage in the MP standards in Precalculus.

Standards for Mathematical Practice Students…	**Examples of each practice in Precalculus**
MP1. Make sense of problems and persevere in	Students expand their repertoire of expressions and functions that can used to solve problems. They grapple with understanding the connection

solving them.	between complex numbers, polar coordinates, and vectors, and reason about them.
MP2. Reason abstractly and quantitatively.	Students understand the connection between transformations and matrices, seeing a matrix as an algebraic representation of a transformation of the plane.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function to model a real-world situation.
MP4. Model with mathematics.	Students apply their new mathematical understanding to real-world problems. Students also discover mathematics through experimentation and examining patterns in data from real-world contexts.
MP5. Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP6. Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of the complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP7. Look for and make use of structure.	Students understand that matrices form an algebraic system in which the order of multiplication matters, especially when solving linear systems using them. They see that complex numbers can be represented by polar coordinates, and that the structure of the plane yields a geometric interpretation of complex multiplication.
MP8. Look for and make use of regularity in repeated reasoning.	Students multiply several vectors by matrices and observe that some matrices give rotations or reflections. They compute with complex numbers and generalize the results to understand the geometric nature of their operations.

MP standard 4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Though the Modeling category has no specific standards listed within it, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Readers will see some standards marked with a star symbol (★) to indicate that they are *modeling standards*, that is, they present an opportunity for applications to real-world modeling situations more so than other standards. Note that this does not preclude other standards from being taught with/through Mathematical Modeling.

Examples of places where specific MP standards can be implemented in the Precalculus standards will be noted in parentheses, with the specific practice standard(s) indicated.

Precalculus Mathematics Content Standards by Conceptual Category

The Precalculus course is organized by conceptual category, domains, clusters, and then standards. Below, the general purpose and progression of the standards included in Precalculus are described according to these conceptual categories. Note that the standards are not listed in an order in which they should be taught.

Conceptual Category: Modeling

Throughout the higher mathematics CA CCSSM, certain standards are marked with a (★) symbol to indicate that they are considered modeling standards. Modeling at this level goes beyond the simple application of previously constructed mathematics to real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: which of the quantities present in this situation are known and unknown? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They will try to use previously derived models (e.g. linear functions) but may find that a new equation or function will apply. They may see that solving an equation arises as a necessity when trying to answer their question, and that oftentimes the equation arises as the specific instance of the knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. This will be a new approach for many teachers and will be challenging to implement, but the effort will produce students who can appreciate that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

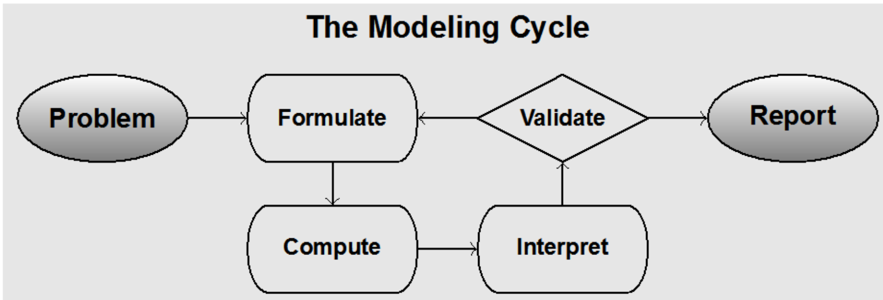

Figure 1: The modeling cycle. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret their results, validate them, and report out.

The reader is encouraged to consult the Appendix, “Mathematical Modeling,” for a further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the functions conceptual category can set the stage for the learning of other standards in Precalculus. At this level, expressions are often viewed as defining outputs for functions, and algebraic manipulations are then performed meaningfully with an eye towards what can be revealed about the function.

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

7d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★

7e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. (+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA★

11. (+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing *f*(*x*) by *f*(*x*) + *k*, *k f*(*x*), *f*(*kx*), and *f*(*x* + *k*) for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.
   - b. (+) Verify by composition that one function is the inverse of another.
   - c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   - d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

While many of the standards in the Interpreting Functions and Building Functions domains appeared in previous courses, students now apply them in the cases of polynomial functions of degree greater than two, more complicated rational functions, the reciprocal trigonometric functions, and inverse trigonometric functions. Students examine end behavior of polynomial and rational functions and learn how to find asymptotes.

Students further their understanding of inverse functions. Whereas before, students only found inverse functions in simple cases (e.g. solving for x, when $f(x) = c$, finding inverses of linear functions), they now explore the relationship between two functions that are inverses of each other, i.e. that f and g are inverses if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, and they may begin to use inverse function notation, expressing g as $g = f^{-1}$. They construct inverse functions by appropriately restricting the domain of a given function and use inverses in contexts. Students in Precalculus understand how a function and its domain and range are related to its inverse function. They realize that finding an inverse function is more than just “switching variables” and solving an equation. They can even find simpler inverses mentally, such as when they reverse the “steps” for the equation $f(x) = x^3 - 1$ to realize that the inverse of f must be $f^{-1}(x) = \sqrt[3]{x+1}$.

Students study parametric functions in Precalculus, understanding that a curve in the plane that might describe the path of a moving object can be represented with such functions. Students also work with polar coordinates and graph polar curves. Connections should be made between polar coordinates and the polar representation of complex numbers (N-CN.4, 5). Students also discover the important role the trigonometric functions play in working with polar coordinates. These standards are new in the typical Precalculus curriculum. Students can investigate these new concepts in modeling situations, such as by recording points on the curve a tossed ball travels along, graphing the points as vectors, and deriving equations for $x(t)$ and $y(t)$. They can also investigate the relationship between the graphs of the sine and cosine as functions of θ on the one hand and the graph of the curve defined by $x(\theta) = \cos\theta, y(\theta) = \sin\theta$ on the other, drawing connections between the two.

Trigonometric Functions **F-TF**

Expand the domain of trigonometric functions using a unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
10. (+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA

In this set of standards, students expand their understanding of the trigonometric functions by connecting properties of the functions to the unit circle, e.g., understanding that since that traveling 2π radians around the unit circle returns one to the same point on the circle, this must be reflected in the graphs of sine and cosine. Students extend their knowledge of finding inverses to doing so for trigonometric functions, and use them in a wide range of application problems.

Students derive the addition and subtraction formulas for sine, cosine and tangent, as well as the half angle and double angle identities for sine and cosine, and make connections between among these. For example, students can derive from the addition formula for cosine ($\cos(x + y) = \cos x \cos y - \sin x \sin y$) the double angle formula for cosine:

$$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$$

Another opportunity for connections arises here, as students can investigate the relationship between these formulas and complex multiplication.

Conceptual Category: Number and Quantity

The Number and Quantity standards in Precalculus represent a culmination in students' understanding of number systems. Students investigate the geometry of the complex numbers more fully and connect it to operations with complex numbers. In addition, students develop the notion of a vector and connect operations with vectors and matrices to transformations of the plane.

The Complex Number System N–CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $(-1 + \sqrt{3}\,i)^3 = 8$ *because* $(-1 + \sqrt{3}\,i)$ *has modulus* 2 *and argument* 120°.
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

As mentioned earlier, complex numbers, polar coordinates, and vectors should all be taught with an emphasis on connections between them. For instance, students connect the addition of complex numbers to the addition of vectors; they also investigate the geometric interpretation of multiplying complex numbers and connect it to polar coordinates using the polar representation.

Vector and Matrix Quantities N–VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., *v*, |*v*|,||*v*||, *v*).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
c. Understand vector subtraction v – w as v + (–w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|v$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|v \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Students investigate vectors as geometric objects in the plane that can be represented by ordered pairs, and matrices as objects that act on vectors. Through working with vectors and matrices both geometrically and quantitatively, students discover that vector addition and subtraction behave according to certain properties, while matrices and matrix operations observe their own set of rules. Attending to structure, students discover with matrices a new set of mathematical objects and operations among them that has a multiplication that is not commutative. They find inverse matrices by hand in 2×2 cases and using technology in other cases. Work with vectors and matrices here sets the stage for solving systems of equations in the Algebra conceptual category.

Conceptual Category: Algebra

In the Algebra conceptual category, Precalculus students work with higher degree polynomials and more complicated rational functions. As always, they attend to the meaning of the expressions they work with, and the expressions they encounter often arise in the context of functions. As in all other Higher Mathematics courses, students work with creating and solving equations, and do so in contexts connected to real-world situations through modeling.

Seeing Structure in Expressions **A-SSE**

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)n$ *as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 – y^4$ *as* $(x^2)^2 – (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 – y^2)(x^2 + y^2)$.

Arithmetic with Polynomials and Rational Expressions **A-APR**

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

By the time students are taking Precalculus, they should have a well-developed understanding the concept of a function. To make work with rational expressions more meaningful, students should be given opportunities to connect rational expressions to rational *functions*, (whose outputs are defined by the expressions). For example, a traditional exercise with rational expressions might have the following form:

$$\textit{Simplify } \frac{200}{x}+\frac{100}{x-10}$$

with the intention that students will find a common denominator and transform the expression into $\frac{300x-2000}{x(x-10)}$. In contrast, students could view the two expressions as defining the outputs of two functions f and g respectively, where $f(x)=\frac{200}{x}$ and $g(x)=\frac{100}{x-10}$. In this case, f could be the function that gives the time it takes for a car to travel 200 miles at an average speed of x miles per hour, while g could be the function that gives the time it takes for the car to travel 100 miles at an average speed of 10 mph less. Students can be asked to consider the domains of the two functions, the domain on which the sum of the two functions defined by $(f+g)(x)=f(x)+g(x)$ makes sense, and what the sum denotes (total time to travel the 300 miles altogether). Furthermore, students can calculate tables of outputs for the two functions using a spreadsheet, add the outputs on the spreadsheet, and then graph the resulting outputs, only to discover that the data fits the graph of the equation $y = \frac{300x-2000}{x(x-10)}$. Finally, if these expressions arise in a modeling context, students can interpret the results of studying these functions and their sum in the real-world context.

Creating Equations **A-CED**

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA★**
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Reasoning with Equations and Inequalities **A-REI**

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Standards A-CED.1-4 appear in most other higher mathematics courses, as they represent common skills involved in working with equations. In Precalculus, students expand these skills into several areas: trigonometric functions, by setting up and solving equations such as $\sin 2\theta = \frac{1}{2}$; parametric functions, by making sense of the equations $x = 2t, y = 3t + 1, 0 \le t \le 10$; and rational expressions, by sketching a rough graph of equations such as $y = \frac{300x-2000}{x(x-10)}$.

Students connect their newfound knowledge of matrices to representing systems of linear equations by matrix multiplication. They can do this in modeling situations, involving payoffs in games, economic quantities, or geometric situations.

Conceptual Category: Geometry

The standards of the Geometry conceptual category also connect back to several other standards found in the Precalculus curriculum. For example, students work with conic sections started, and opportunities to view conic sections as parametric functions provide a rich ground for studying such functions (F-IF.10).

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula *A* = 1/2 *ab* sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation in standard form; identify whether the graph of the equation is a circle, parabola, ellipse, or hyperbola, and graph the equation. CA

Students continue their study of trigonometric functions by discovering that they can be introduced into general triangles using appropriate auxiliary lines. The relationships that they give rise to then result in the Laws of Sines and Cosines in general cases. Students can derive these laws and use them to solve problems, and they connect the relationships they describe to the geometry of vectors.

Precalculus Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Arithmetic with Polynomials and Rational Expressions

- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build new functions from existing functions.

Trigonometric Functions

- Expand the domain of trigonometric functions using a unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Precalculus

Conceptual Category: Number and Quantity

<u>The Complex Number System N –CN</u>

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $(-1 + \sqrt{3}\,i)^3 = 8$ *because* $(-1 + \sqrt{3}\,i)$ *has modulus* 2 *and argument* 120°.
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

<u>Vector and Matrix Quantities N –VM</u>

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, $|v|$, $||v||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. Understand vector subtraction v – w as v + (–w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   b. Compute the magnitude of a scalar multiple $\boldsymbol{cv}$ using $||\boldsymbol{cv}|| = |c|v$. Compute the direction of $\boldsymbol{cv}$ knowing that when $|c|v \neq 0$, the direction of $\boldsymbol{cv}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Conceptual Category: Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)n$ *as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Arithmetic with Polynomials and Rational Expressions A-APR

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A -CED

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA★**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Reasoning with Equations and Inequalities A -REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Conceptual Category: Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

7d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★

7e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. (+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA★

11. (+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.
   - b. (+) Verify by composition that one function is the inverse of another.
   - c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   - d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Trigonometric Functions F-TF

Expand the domain of trigonometric functions using a unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

10. (+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. **CA** ★

Conceptual Category: Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation in standard form; identify whether the graph of the equation is a circle, parabola, ellipse, or hyperbola, and graph the equation. CA