Mathematics III

Introduction

The standards in the integrated Mathematics III course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. Students expand their repertoire of functions to include polynomial, rational, and radical functions. They expand their study of right triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses of the Integrated Pathway follow the structure began in the K-8 standards of presenting mathematics as a coherent subject, mixing standards from various conceptual categories.

The content of the course will be expanded upon below according to these conceptual categories, but teachers and administrators alike should note that the standards are not topics to be checked off a list during isolated units of instruction, but rather content that should be present throughout the school year through rich instructional experiences. In addition, the standards should not necessarily be taught in the order in which they appear here, but rather in a coherent manner.

What Students learn in Mathematics III

Overview

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. The work on polynomial expressions culminates with the Fundamental Theorem of Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data— including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

Finally, in Mathematics III, students extend their understanding of modeling: they identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” is one of the main themes of this course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers we can add, subtract and multiply, etc. Subsequently, polynomials can be extended to rational expressions, analogous to the rational numbers.
- The understandings that students have developed with linear, exponential and quadratic functions are extended to considering a much broader range of classes of functions.
- In statistics, students begin to look at the role of randomization in statistical design.

Connecting Standards for Mathematical Practice and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics in* the classroom and, to the extent possible, content instruction should include attention to appropriate practice standards. The table below gives examples of how students can engage in the MP standards in Mathematics III.

Standards for Mathematical Practice Students…	**Examples of each practice in Mathematics III**
MP1. Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real world problems. They approach complex mathematics problems and break them down into smaller-sized chunks and synthesize the results when presenting solutions.
MP2. Reason abstractly and quantitatively.	Students deepen their understanding of variable, for example, by understanding that changing the values of the parameters in the expression $A \sin(Bx + C) + D$ has consequences for the graph of the function. They

	interpret these parameters in a real world context.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function to model a real world situation.
MP4. Model with mathematics.	Students apply their new mathematical understanding to real world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and examining patterns in data from real world contexts.
MP5. Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP6. Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of the complex numbers. They pay attention to units in real world problems and use unit analysis as a method for verifying their answers.
MP7. Look for and make use of structure.	Students understand the polynomials and rational numbers as sets of mathematical objects that have certain operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP8. Look for and make use of regularity in repeated reasoning.	Students observe patterns in geometric sums, e.g. that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written: $1 = 2^1 - 1; 1 + 2 = 2^2 - 1;$ $1 + 2 + 4 = 2^3 - 1; 1 + 2 + 4 + 8 = 2^4 - 1$, and use this observation to make a conjecture about any such sum.

MP standard 4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Though the Modeling category has no specific standards listed within it, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Readers will see some standards marked with a star symbol (★) to indicate that they are *modeling standards*, that is, they present an opportunity for applications to real-world modeling situations more so than other standards.

Examples of places where specific MP standards can be implemented in the Mathematics III standards will be noted in parentheses, with the specific practice standard(s) indicated.

Mathematics III Content Standards by Conceptual Category

The Mathematics III course is organized by conceptual category, domains, clusters, and then standards. Below, the overall purpose and progression of the standards included in Mathematics III are described according to these conceptual categories. Note that the standards are not listed in an order in which they should be taught. Standards that are considered to be new to secondary grades teachers will be discussed in more depth than others.

Conceptual Category: Modeling

Throughout the higher mathematics CA CCSSM, certain standards are marked with a (★) symbol to indicate that they are considered modeling standards. Modeling at this level goes beyond the simple application of previously constructed mathematics to real world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real world situation and challenged to ask a question, all sorts of new issues arise: which of the quantities present in this situation are known and unknown? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They will try to use previously derived models (e.g. linear functions) but may find that a new formula or function will apply. They may see that solving an equation arises as a necessity when trying to answer their question, and that oftentimes the equation arises as the specific instance of the knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. This will be a new approach for many teachers and will be challenging to implement, but the effort will produce students who can appreciate that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

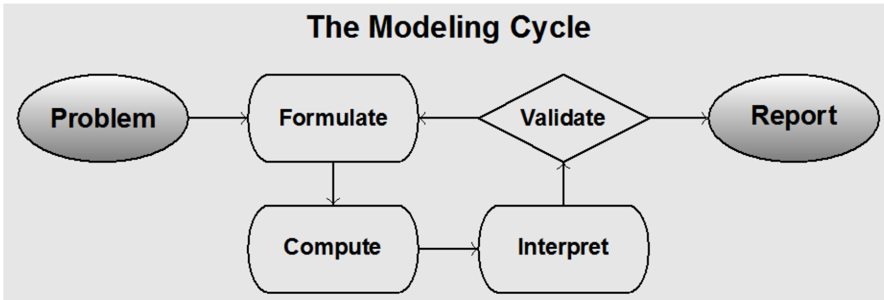


Figure 1: The modeling cycle. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret their results, validate them, and report out.

Throughout the Mathematics III chapter, the examples given will be framed as much as possible as modeling situations, to serve as illustrations of the concept of mathematical modeling. The big ideas of rational functions, graphing, solving equations, and rates of change will be explored through this lens. The reader is encouraged to consult the Appendix, "Mathematical Modeling," for a further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can serve as motivation for studying the standards in the other conceptual categories. Students have already worked with equations wherein one is asked to "solve for x" as a search for the input of a function f that gives a specified output. Solving the equation amounts to undoing the work of the function. In Mathematics III, the types of functions students encounter have new properties. Students saw how quadratic functions exhibited behavior different from linear and exponential functions; now, they investigate polynomial and rational functions and the trigonometric functions in greater generality. Once again, new techniques for solving the equations that arise must be discovered. Students see how rational functions can model certain real world phenomena, in particular in instances of inverse variation ($x \cdot y = k, k$ a constant), and how trigonometric functions can model periodic phenomena. In general, functions describe how two quantities are related in a precise way, and can be used to make predictions and generalizations, keeping true to the emphasis on modeling in Higher Mathematics. Students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary (The University of Arizona Progressions Documents for the Common Core Math Standards [Progressions], Functions 2012, 7).

Interpreting Functions **F-IF**

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

As in Mathematics II, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots and other characteristics of functions is becoming more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (F-IF.4-9). The problem below exemplifies some of these standards. (Note that only sine, cosine and tangent are treated in Mathematics III.)

Example. *The Juice Can.* Suppose we wanted to know the minimal surface area of a cylindrical can of a fixed volume. Here, we consider the surface area in units cm^2, the radius in units cm, and the volume to be fixed at 355 ml = 355 cm^3. One can find the surface area of this can as a function of the radius:

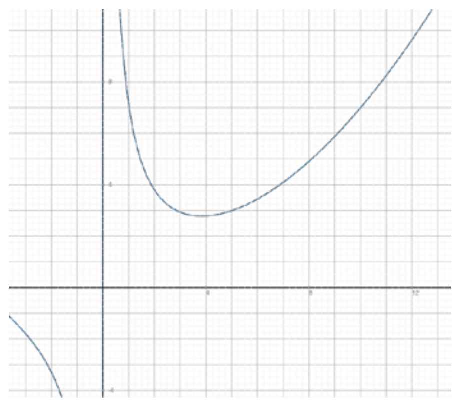
$$S(r) = \frac{2(355)}{r} + 2\pi r^2.$$

(See *The Juice Can Equation* example in the Algebra conceptual category.) This representation allows us to examine several things.

First, a table of values will give a hint at what the minimal surface area is. The table shown lists several values for S based on r:

r (cm)	S (cm^2)
0.5	1421.6
1.0	716.3
1.5	487.5
2.0	380.1
2.5	323.3
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	299.0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

The data suggests that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 cm. Successive approximation using values of r between these values will yield a better estimate. But how can we be sure that the minimum is truly located here? A graph of $S(r)$ can give us a hint:



Furthermore students can deduce that as r gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller and that the reverse is true as r grows larger, so that there is truly a minimum somewhere in the interval $[3.5, 4.5]$. (F-IF.4, F-IF.5, F-.IF.7-9)

Graphs help us reason about rates of change of functions (F.IF.6). Students learned in grade eight that the *rate of change* of a linear function is equal to the slope of its graph. And because the slope of a line is constant, the phrase “rate of change” is clear for linear functions. For nonlinear functions, however, rates of change are not constant, and so we talk about average rates of change over an interval. For example, for the function g defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points $(2, 4)$ and $(5, 25)$ on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (Progressions, Functions 2012, 9). Students could investigate similar rates of change over intervals for the Juice Can problem shown previously.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Students in Mathematics III develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (F-BF.1). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations involving cooling or heating involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is $70°$ and a cup of tea is made with boiling water at a temperature of $212°$, a student can express the function describing the temperature as a function of time by using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, leading to a function of the form:

$$T(t) = 70 + 142e^{-kt}.$$

Students might determine the constant k experimentally. (MP.4, MP.5)

In F-BF.4a, students learn that some functions have the property that an input can be recovered from a given output, i.e., the equation $f(x) = c$ can be solved for x, given that c lies in the range of f. They understand that this is an attempt to “undo” the function, or to “go backwards.” Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard F-LE.4 described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, on so this standard should be treated carefully as preparation for a deeper understanding.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.★ [Logarithms as solutions for exponentials.]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA★

Students have worked with exponential models in Mathematics II. Based on the graph of the exponential $f(x) = b^x$, we can deduce that this function has an inverse, called *the logarithm to the base* b, and denoted by $g(x) = \log_b x$. The logarithm has the property that $\log_b x = y$ if and only if $b^y = x$. Students find logarithms with base b equal to 2, 10, or e, by hand and using technology (MP.5). Students may be encouraged to explore the properties of logarithms, such as that $\log_b xy = \log_b x + \log_b y$, and to connect these properties to those of exponents (e.g. the previous property comes from the fact that the logarithm is essentially an exponent, and that $b^{n+m} = b^n \cdot b^m$). Students solve problems involving exponential functions and logarithms and express their answers using logarithm notation (F-LE.4). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; opportunities for instruction should emphasize this relationship.

Trigonometric Functions **F-TF**

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

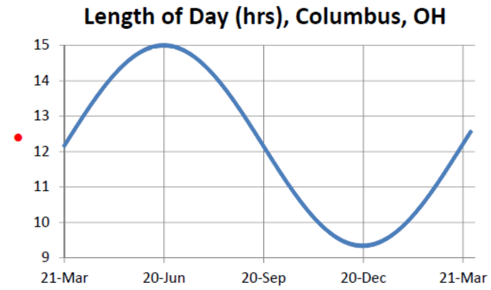
Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

In this set of standards, students expand on their understanding of the trigonometric functions first developed in Mathematics II. At first, the trigonometric functions apply only to angles in right triangles, i.e. $\sin\theta$, $\cos\theta$, and $\tan\theta$ only make sense for $0 < \theta < \frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, we see that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of θ that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, we take $\cos\theta$ to be the x-coordinate of the point corresponding to this rotation, and $\sin\theta$ to be the y-coordinate of this point. This interpretation of sine and cosine immediately yield the Pythagorean Identity: that $\cos^2\theta + \sin^2\theta = 1$. This basic identity yields others through algebraic manipulation, and allows one to find values of other trigonometric functions for a given θ if one of them is known (F-TF.1, 2, 8).

The graphs of the trigonometric functions should be explored with attention to the connection between the unit circle representation of the trigonometric functions and their properties, e.g. to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeroes, etc. In standard F-TF.5, students use trigonometric functions to model periodic phenomena. Connected to standard F-BF.3 (families of functions), they begin to understand the relationship between the parameters appearing in the general cosine function $f(x) = A \cdot \cos(Bx - C) + D$ (and sine function) and the graph and behavior of the function (e.g. amplitude, frequency, line of symmetry). In addition, they use their understanding of inverse functions to explore the inverse sine, cosine and tangent functions at a basic level. Of importance here is the understanding that a function is well-defined only when its domain is specified. For example, the general sine function $\sin x$, defined for all real numbers x does not have an inverse, while the function $s(x) = \sin x$, defined only for values of x such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, *does* have an inverse function.

Example (Progressions, Functions 2012, 19): *Modeling Daylight Hours.* By looking at data for length of days in Columbus, OH, students see that day length is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the different of the maximum and minimum. We set $A = 12.17$ and $B = 2.83$ as approximations of these values. With some support, students determine that for the period to be 365 days (per cycle), $C = 2\pi/365$ and garden, i.e. when there are at least 7 hours of midday sunlight, students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$, and find that May 1 and August 10 bookend this interval of time.

if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$. Thus, $f(t) = 12.17 + 2.83 \sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t the day of the year from March 21. Considering questions such as when to plant a	• or for the frequency to be $\frac{1}{365}$ cycles/day  Students can investigate many other trigonometric modeling situations such as simple predator-prey models, sound waves, and noise cancellation models.

Conceptual Category: Number and Quantity

The Complex Number System **N-CN**

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N.CN.9 to higher degree polynomials.]

8. (+) Extend polynomial identities to the complex numbers.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

In standards N-CN.8 and N-CN.9, students continue to work with complex numbers as solutions to polynomial equations, building off of their work with quadratics started in Mathematics II. For example, when seen in the light of the Mathematics III algebra standards (e.g., A-APR.2), students can find roots of equations such as $x^3 + 5x^2 + 8x + 6 = 0$. They experiment using the remainder theorem and find that $x + 3$ is a root, since the polynomial expression evaluated at $x = -3$ is 0. Using polynomial long division or other factoring techniques, students find that $x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2)$. They use the quadratic formula to find the roots of the quadratic, $\{-1 + i, -1 - i\}$, and they write

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2).$$

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x - (-1 + i))(x - (-1 - i))$$

Experimentation with examples of such polynomials, and an understanding that the quadratic formula always yields solutions to a quadratic equation helps students understand the Fundamental Theorem of Algebra (N-CN.9).

Conceptual Category: Algebra

Along with the Number and Quantity standards in Mathematics III, the Algebra domain standards in Mathematics III develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. In a similar way that rational numbers extend the arithmetic of integers by allowing division by all numbers except 0, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme that arises is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Seeing Structure in Expressions **A-SSE**

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

In Mathematics III, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by “chunking”, that is, by viewing parts of an expression as a single entity (A-SSE.1, 2). For example, their facility with using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

In addition, with their new understanding of complex numbers, this can be factored further into $x^4 - y^4 = (x + iy)(x - iy)(x + y)(x - y)$. In a Physics course, students may encounter an expression such as $L_0\sqrt{1 - \frac{v^2}{c^2}}$, which arises in the theory of special relativity. They can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v = 0$ and equal to 0 when $v = c$—and furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (Progressions, Algebra 2012, 4).

By examining the sum of a finite geometric series, students can look for a pattern to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{(1-r)}$. They may derive the formula, either with Proof by Mathematical Induction, or by other means (A-SSE.4).

Example. *Sum of a Geometric Series.* Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (MP.8). Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments like retirement accounts, finding total lottery payout prizes, and more (MP.4).	In general, a finite geometric series has the form: $\sum_{k=0}^{n} ar^k = a(1 + r + r^2 + \cdots + r^{n-1} + r^n).$ If we denote by S the sum of this series, then some algebraic manipulation shows that $S - rS = a - ar^{n+1}$ Applying the distributive property to the common factors, and solving for S, shows that $S(1 - r) = a(1 - r^{n+1}),$ so that $S = \frac{a(1 - r^{n+1})}{1 - r}.$

Students develop the ability to flexibly see expressions such as $A_n = A_0\left(1 + \frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret

$$A_1 + A_2 + \cdots + A_{12} = 100\left(1+\frac{.15}{12}\right)^1 + 100\left(1+\frac{.15}{12}\right)^2 + \cdots 100\left(1+\frac{.15}{12}\right)^{12}$$

as a type of geometric series that would calculate the total value in an investment account at the end of one year if we deposited $100 at the beginning of each month (MP.2, MP.4, MP.7). They apply the formula for the sum of a geometric series to find this total.

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle[1]
★

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

[1] The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument.

In Mathematics III, students continue developing their understanding of the set of polynomials as a system analogous to the set of integers that exhibits certain properties and they explore the relationship between the factorization of polynomials and the roots of a polynomial (A-APR.1-3). It is shown that when we divide a polynomial $p(x)$ by $(x-a)$, we are writing $p(x)$ in the following way:

$$p(x) = q(x) \cdot (x-a) + r,$$

where r is a constant. This can be done by inspection or by polynomial long division (A-APR.7). It follows that $p(a) = q(a) \cdot (a-a) + r = q(a) \cdot 0 + r = r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a) = 0$. This result is generally known as the Remainder Theorem (A-APR.2), and provides an easy check to see if a polynomial has a given linear factor. This topic should not be simply reduced to “synthetic division,” which reduces the theorem to a method of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (MP.3) (Progressions, Algebra 2012, 7).

Students use the zeroes of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (A-APR.3). The notion that the polynomials can be used to approximate other functions is important in higher mathematics courses such as Calculus, and students can get a start here. Standard A-APR.3 is the first step in a progression that can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane (Progressions Algebra 2012, 7).

In addition, polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers (A-APR.4). For example, students can explore the sequence of squares 1, 4, 9, 16, 25, 36, … and notice the differences between them—3, 5, 7,9 , 11—are consecutive odd integers. This mystery is explained by the polynomial identity $(n+1)^2 - n^2 = 2n + 1$, which can be justified using pictures (Progressions, Algebra 2012, 6).

In Mathematics III, students explore rational functions as a system analogous to the rational numbers. They see rational functions as useful for describing many real-world situations, for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance d_0, and obtaining $r = \frac{d_0}{t}$. Now students see that any two polynomials can be divided in much the same way as with numbers (provided the divisor is not 0). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the special form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials in x. They should evaluate various rational expressions for many values of x, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. Developing an understanding of the behavior of rational expressions in this way helps students see them as functions, and sets the stage for working with simple rational functions in the Functions domain.

Example. *The Juice Can.* If someone wanted to investigate the shape of a juice can of minimal surface area, they could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the formula:

$$S = 2\pi rh + 2\pi r^2,$$

since the two circular bases of the can contribute $2\pi r^2$ units of surface area, while the outside surface of the can contributes an area in the shape of a rectangle with length the circumference of the base, $2\pi r$, and height h. Since the volume is fixed, we can find h in terms of r: $h = \frac{V_0}{\pi r^2}$, and substitute into the equation for the surface area:

$$\begin{aligned} S &= 2\pi r \cdot \frac{V_0}{\pi r^2} + 2\pi r^2 \\ &= \frac{2V_0}{r} + 2\pi r^2. \end{aligned}$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed. (See also A-CED.4.)

In addition, students are able to rewrite rational expressions in the form $a(x) = q(x) \cdot b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary, e.g. to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in manipulating symbols properly, instruction should focus on the characteristics of rational functions that can be understood by rewriting in the ways above; e.g., rates of growth, approximation, roots, axis-intersections, asymptotes, end behavior, etc.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA**★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

In Mathematics III, students work with all available types of functions to create equations, including root functions (A-CDE.1). While functions used in A-CED.2, 3, and 4 will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Mathematics I and II. For example, knowing how to find the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. For an example of standard A-CED.4, see the *Juice Can* problem earlier in this section.

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Students extend their equation solving skills to those that involve rational expressions and radical equations and they make sense of extraneous solutions when they arise (A-REI.2). In particular, students understand that when solving equations, the flow in reasoning is generally forward, in the sense that we assume a number x is a solution of the equation and then find a list of possibilities for x. But not all steps in this process are reversible, e.g. while it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (Progressions, Algebra 2012, 10). Thus students understand that some steps are reversible and some are not, and anticipate extraneous solutions. In addition, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (A-REI.11), and understand that some equations can only be solved approximately with the tools they possess.

Conceptual Category: Geometry

Similarity, Right Triangles, and Trigonometry **G-SRT**

Apply trigonometry to general triangles.

9. (+) Derive the formula *A* = 1/2 *ab* sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Students advance their knowledge of right triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is a and the height is $b \cdot \sin C$, they derive a general formula for the area of any triangle $A = \frac{1}{2} ab \sin C$ (G-SRT.9). In addition, students use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines first in only acute triangles, and use these and other relationships to solve problems (G-SRT.10-11). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to $90°$ to fully discuss Laws of Sine and Cosine, though full unit circle trigonometry need not be discussed in this course.

Geometric Measurement and Dimension G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

This set of standards is rich with opportunities for students to apply modeling (MP.4) with geometric concepts. The implementation of these standards should not be limited to the end of any course simply because they are later in the sequence of standards; they should be employed throughout the geometry curriculum of the course. In standard G-MG.1, students use geometric shapes, their measures, and their properties to describe objects. This standard can involve 2- and 3-dimensional shapes, and is not relegated to simple applications of formulas. In standard G-MG.3, students solve design problems by modeling with geometry.

Example (Illustrative Mathematics 2013). *Ice Cream Cone.* You have been hired by the owner of a local ice cream parlor to assist in his company's new venture. The company will soon sell its ice cream cones in the freezer section of local grocery stores. The manufacturing process requires that the ice cream cone be wrapped in a cone-shaped	b. Use your sketch to help you develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone given its base had a radius of length, r, and a slant height, s. c. Using measurements of the radius of the base and slant height of your cone, and your

paper wrapper with a flat circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice cream cone or the dimensions of a real ice cream cone to complete the following tasks. **Some possible questions:** a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.	equation from the previous step, find the surface area of your cone. d. The company has a large rectangular piece of paper that measures 100 cm by 150 cm. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this one piece of paper. Explain your estimate. Solutions can be found at *illustrativemath.org*

Expressing Geometric Properties with Equations **G-GPE**

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola, and graph the equation. [In Mathematics III, this standard addresses circles and parabolas only.] **CA**

Students further their understanding of the connection between algebra and geometry by applying the definition of circles and parabolas to derive equations for them, and then deciding whether a given quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$ represents a circle or parabola.

Conceptual Category: Statistics and Probability

Interpreting Categorical and Quantitative Data **S-ID**

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. In Mathematics III, students build on their understanding of data distributions to help see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (S-ID.4). In addition, they can learn through examples the *empirical rule*, that for a normally distributed data set, 68% of the data lies within one standard deviation of the mean, and that 95% are within two standard deviations of the mean.

Example. *The Empirical Rule.* Suppose that SAT mathematics scores for a particular year are approximately normally distributed with a mean of 510 and a standard deviation of 100. a. What is the probability that a randomly selected score is greater than 610? b. Greater than 710? c. Between 410 and 710? d. If a students' score is 750, what is the students' percentile score (the proportion of scores below 750)? e.	**Solutions:** a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32 or 0.16. The calculator gives 0.1586. b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05 or 0.025. The calculator gives 0.0227. c. The area under a normal curve from one standard deviation below the mean to two standard deviations above is about 0.815. The calculator gives 0.8186. d. Either using the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4) the calculator gives 0.9918. f.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★

5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Students in Mathematics III move beyond analyzing data to making sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (S-IC.1) (Progressions, High School Statistics and Probability 2012). By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (S-IC.2). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment (CCSSI 2010). In standards S-IC.4 and 5, the focus should be on the variability of results from experiments—that is, focused on statistics as a way of dealing with, not eliminating, inherent randomness. Given that standards S-IC.1-6 are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example (Progressions, High School Statistics and Probability 2012). *Estimating a Population*	If we simulate this sampling situation (MP.4) using a graphing calculator or spreadsheet software

Proportion. Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 agree, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the *true* proportion of homeowners who favor the tax? How can we tell?	under the assumption that the true proportion is 50%, then she can get an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of .125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

As in Mathematics II, students apply probability models to make and analyze decisions. In Mathematics III, this skill is extended to more complex probability models, including situations such as those involving quality control, or diagnostic tests that yield both false positive and false negative results. See the “High School Progression on Statistics and Probability” for more explanation and examples: http://ime.math.arizona.edu/progressions/.

Mathematics III Overview

Number and Quantity

The Complex Number System

- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making

(+) Indicates additional mathematics to prepare students for advanced courses

Mathematics III

Number and Quantity

The Complex Number System
N-CN

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N.CN.9 to higher degree polynomials.]

8. (+) Extend polynomial identities to the complex numbers.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle[2]
   ★

[2] The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument.

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA**★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

1. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★ [Logarithms as solutions for exponentials]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA★

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Geometry

Similarity, Right Triangles, and Trigonometry — G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola, and graph the equation. [In Mathematics III, this standard addresses only circles and parabolas.] **CA**

Geometric Measurement and Dimension — G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

Statistics and Probability

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★