Mathematics II

Introduction

The focus of Mathematics II is on quadratic expressions, equations, and functions, and comparing their characteristics and behavior to those of linear and exponential relationships from Mathematics I. The need for extending the set of rational numbers arises and real and complex numbers are introduced. The link between probability and data is explored through conditional probability and counting methods, including their use in making and evaluating decisions. The study of similarity leads to an understanding of right triangle trigonometry and connects to quadratics through Pythagorean relationships. Circles, with their quadratic algebraic representations, round out the course. The courses of the Integrated Pathway follow the structure began in the K-8 standards of presenting mathematics as a coherent subject, mixing standards from various conceptual categories.

The standards in the integrated Mathematics II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course will be expanded upon below according to these conceptual categories, but teachers and administrators alike should note that the standards are not topics to be checked off a list during isolated units of instruction, but rather content that should be present throughout the school year through rich instructional experiences. In addition, the standards should not necessarily be taught in the order in which they appear here, but rather in a coherent manner.

What Students learn in Mathematics II

Overview

In Mathematics II, students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system can be extended so that solutions exist, analogous to the way in which extending the whole numbers to the negative numbers allows $x + 1 = 0$

to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. When quadratic equations do not have real solutions, students learn that that the graph of the related quadratic function does not cross the horizontal axis. Students expand their experience with functions to include more specialized functions—absolute value, step, and other piecewise-defined functions.

Students focus on the structure of expressions, writing equivalent expressions to clarify and reveal aspects of the quantities they represent. Students create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

Building on probability concepts introduced in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions.

Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. It is in this course that students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They explore a variety of formats for writing proofs.

In Mathematics II students prove basic theorems about circles, chords, secants, tangents, and angle measures. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center, and the equation of a parabola with vertical axis when given an equation of its horizontal directrix and the coordinates of its focus. Given an equation of a circle, students draw the graph in the coordinate plane, and apply techniques for solving quadratic equations to determine intersections between lines and circles or a parabola and between two circles. Students develop informal arguments justifying common formulas for circumference, area, and volume of geometric objects, especially those related to circles.

Examples of Key Advances from Mathematics I

Students extend their previous work with linear and exponential expressions, equations, systems of equations and inequalities to quadratic relationships.

- A parallel extension occurs from linear and exponential functions to quadratic functions, where all students begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships, particularly around notions of similarity.

Connecting Standards for Mathematical Practice and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics in* the classroom and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the standards for mathematical practice.

Connecting content and practices happens in the context of *working on problems*; the very first MP standard is to make sense of problems and persevere in solving them. The table below gives examples of how students can engage in the MP standards in Mathematics II.

Standards for Mathematical Practice Students…	**Examples of each practice in Mathematics II**
MP1. Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between quadratic functions and linear and exponential functions studied previously. They make diagrams of geometric problems to help them make sense of the problems.
MP2. Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students construct proofs of geometric theorems based on congruence criteria of triangles. They understand and explain the definition of radian measure.
MP4. Model with mathematics.	Students apply their mathematical understanding of quadratic functions to real-world problems. Students also discover mathematics through experimentation and examining patterns in data from real-world contexts.
MP5. Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.
MP6. Attend to precision.	Students begin to understand that a *rational number* has a specific definition, and that *irrational numbers* exist. They make use of the definition of *function* when deciding if an equation can describe a function by asking, “Does every input value have exactly one output value?”
MP7. Look for and make use of structure.	Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5 + (x - 2)^2$ takes the form of “5 plus ‘something’ squared”, and so that expression can be no smaller than 5.
MP8. Look for and express regularity in repeated reasoning.	Students notice that consecutive numbers in the sequence of squares $1,4,9,16,25$ always differ by an odd number. They use polynomials to represent this interesting finding by expressing it as $(n + 1)^2 - n^2 = 2n + 1$.

MP standard 4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Though the Modeling category has no specific standards listed within it, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Readers will see some standards marked with a star symbol (★) to indicate that they are *modeling standards*, that is, they present an opportunity for applications to real-world modeling situations more so than other standards. Modeling with Mathematics is a theme in all higher math courses. Modeling in higher mathematics centers on problems arising in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the current course.

Examples of places where specific MP standards can be implemented in the Mathematics II standards will be noted in parentheses, with the specific practice standard(s) indicated.

Mathematics II Content Standards by Conceptual Category

The Mathematics II course is organized by conceptual category, domains, clusters, and then standards. Below, the overall purpose and progression of the standards included in Mathematics II are described according to these conceptual categories. Note that the standards are not listed in an order in which they should be taught. Standards that are considered to be new to secondary grades teachers will be discussed in more depth than others.

Conceptual Category: Modeling

Throughout the higher mathematics CA CCSSM, certain standards are marked with a (⋆) symbol to indicate that they are considered modeling standards. Modeling at this level goes beyond the simple application of previously constructed mathematics to real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: which of the quantities present in this situation are known and unknown? Students need to decide on a solution path; sometimes that path may need to be revised. They will make use of tools such as calculators, dynamic geometry software, or spreadsheets. They will try to use previously derived models (e.g. linear functions) but may find that a new formula or function will apply. They may see that solving an equation arises as a necessity when trying to answer their question, and that oftentimes the equation arises as the specific instance of the knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. This will be a new approach for many teachers and will be challenging to implement, but the effort will produce students who can appreciate that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

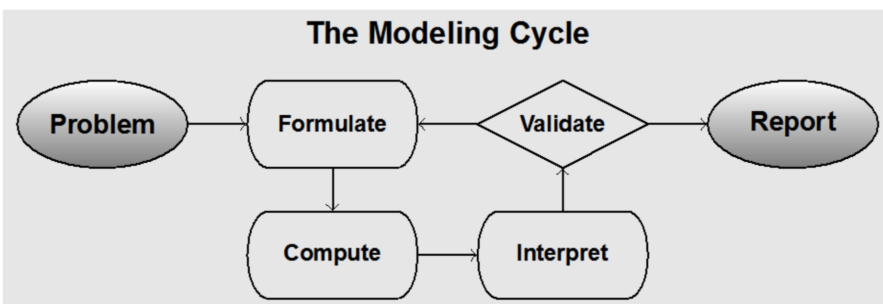


Figure 1: The modeling cycle. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret their results, validate them, and report out.

Throughout the Mathematics II chapter, the examples given will be framed as much as possible as modeling situations, to serve as illustrations of the concept of mathematical modeling. The big ideas of quadratic functions, graphing, solving equations, and rates of change will be explored through this lens. The reader is encouraged to consult the Appendix, "Mathematical Modeling," for a further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can serve as motivation for studying the standards in the other Mathematics II conceptual categories. Students have already worked with equations wherein one is asked to "solve for x" as a search for the input of a function f that gives a specified output; solving the equation amounts to undoing the work of the function. In Mathematics II, the types of functions students encounter have new properties. For example, while linear functions showed constant additive change and exponential functions showed constant multiplicative change, quadratic functions exhibit different change and can be used to model new situations. New techniques for solving equations need to be carefully constructed as extraneous solutions can sometimes arise, or no real-number solutions may exist. In general, functions describe how two quantities are related in a precise way, and can be used to make predictions and generalizations, keeping true to the emphasis on modeling in higher mathematics. The core question when investigating functions in the CA CCSSM is: "Does each element of the domain correspond to exactly one element of the range?" (The University of Arizona Progressions Documents for the Common Core Math Standards [Progressions], Functions 2012, 8)

Interpreting Functions **F-IF**

Interpret functions that arise in applications in terms of the context. [Quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, *and* $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Standards F.IF.4-9 deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (MP.6). In standards F.IF.7-9 specifically, students represent functions with graphs and identify key features of the graph. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context. For instance, students may easily see that the function $f(x) = 3x^2 + 9x + 6$ crosses the y-axis at $(0,6)$ since the terms involving x are simply 0 when $x = 0$, but they then factor the expression defining f to obtain $f(x) = 3(x + 2)(x + 1)$, revealing that the function crosses the x-axis at $(-2, 0)$ and $(-1, 0)$ because those points correspond to where $f(x) = 0$ (MP.7). In Mathematics I, students work with linear, exponential and quadratic functions, and are expected to develop fluency with these types of functions, including the ability to graph them by hand.

Students work with functions that model data and with choosing an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots and other characteristics of functions is becoming more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (F-IF.4-9).

Example (Adapted from Illustrative Mathematics 2013). *Population Growth.* The approximate United States Population measured each decade starting in 1790 up through 1940 can be modeling by the function

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}.$$

a. According to this model for the U.S. population, what was the population in the year 1790?

b. According to this model, when did the population first reach 100,000,000? Explain.

c. According to this model, what should be the population of the U.S. in the year 2010? Find a prediction of the U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Quadratic and exponential]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. ★

Build new functions from existing functions. [Quadratic, absolute value]

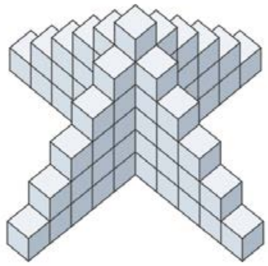
3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$.

Students in Mathematics II develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (F-BF.1). The problem below illustrates reasoning with functions students are expected to develop in this standard.

Example. *The Skeleton Tower*.[1]

The tower pictured measures 6 cubes high.

(a) How many cubes are needed to build this tower? (Organize your counting so others can follow your reasoning.)



(b) How many cubes would be needed to build a tower just like this one but 12 cubes high? Justify your reasoning.

(c) Find a way to calculate the number of cubes needed to build a tower like this that is n cubes high.

Solution:

a. The top layer has a single cube. The layer below has one cube beneath the top cube plus 4 new ones making a total of 5. The third layer below has cubes below these 5 plus 4 new ones to make 9. Continuing to add four each time gives a total of $1+5+9+13+17+21=66$ cubes in the skeleton tower with six layers.

b. Building upon the reasoning established in (a), the number of cubes in the bottom (12th) layer will be $1+4\times 11$ since it is 11 layers below the top. So for this total, we need to add $1+5+9+\cdots+45$. One way to do this would be to simply add the numbers. Another method is the Gauss method: We rewrite the sum backward as $45+41+37+\cdots+1$. Now if this sum is placed below the previous sum we can see that each pair of addends one above the other sums to 46. There are 12 columns so the answer to this problem is half of $12\times 46=552$ or 276.

c. Let $f(n)$ be the number of cubes in the nth layer counting down from the top. Then $f(1)=1$, $f(2)=5$, $f(3)=9$, and so on. Since each term is obtained from the previous one by adding 4, in general, we have $f(n)=4(n-1)+1$. The total for n layers in the tower is thus: $1+5+9+\cdots+f(n)=1+5+9+\cdots+(4(n-1)+1)$. If we use the method of problem (b) here, twice this sum will be equal to $n\cdot(4(n-1)+2)$ and so the general solution for the number of cubes in a skeleton tower with n layers is $n(4(n-1)+2)/2=n(2n-1)$.

For standard F-BF.3, students can make good use of graphing software to investigate the effects of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for different types of functions. For example, starting with the simple quadratic function $f(x)=x^2$, students see the relationship between the transformed functions $f(x)+k, kf(x), f(kx),$ and $f(x+k)$ and the vertex-form of a general quadratic, $f(x)=a(x-h)^2+k$. They understand the notion of a *family of functions*, and characterize

[1] For another solution, see www.illustrativemathematics.org.

such function families based on their properties. In keeping with the theme of the input-output interpretation of a function, students should work towards developing an understanding of the effect on the output of a function under certain transformations, such as in the table below:

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x

Such understandings can help in seeing the effect of transformations on the graph of a function, and in particular, can aid in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable (e.g. the graph of $y = f(x + 2)$ is the graph of f shifted 2 units to the left, not to the right) (Progressions, Functions 2012, 7). These ideas will be explored further with trigonometric functions (F-TF.5) in Mathematics III.

In standard F-BF.4a, students learn that some functions have the property that an input can be recovered from a given output, i.e., the equation $f(x) = c$ can be solved for x, given that c lies in the range of f. For example, a student might solve the equation $C = \frac{9}{5}F + 32$ for F. The student starts with this formula showing how Celsius temperature is a function of Fahrenheit temperature and, by solving for F, finds the formula for the inverse function. This is a contextually appropriate way to find the expression for an inverse function, in contrast with the practice of simply swapping x and y in an equation and solving for y.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Include quadratic.]

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

6. Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA★

In Mathematics II, students continue their investigation of exponential functions by comparing them with linear and quadratic functions, observing that they will always eventually grow larger than any polynomial function. In standard F-LE.6, students experiment with quadratic functions and discover their uses in representing real-world phenomena, such as projectile motion. A simple activity that involves tossing a ball and recording its height with video as it rises and falls can reveal the height as a function of time to be approximately quadratic. Afterwards, students can derive a quadratic expression that determines the height of the ball at time t using a graphing calculator or other software, and they can compare the values of the function with their data.

Trigonometric Functions F-TF

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Standard F-TF.8 is closely linked with standards G-SRT.6-8 but is included here as a property of the trigonometric functions sine, cosine and tangent. Students use the Pythagorean identity to find the output of a trigonometric function at given angle θ when the output of another trigonometric function is known.

Conceptual Category: Number and Quantity

Number and Quantity

The Real Number System N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

In grade eight, students encountered some examples of irrational numbers, such as π and $\sqrt{2}$ (or $\sqrt{p}$ for p a nonsquare number). In Mathematics II, students extend this understanding beyond the fact that there are numbers that are not rational; they begin to understand that the rational numbers form a closed system themselves. Students have witnessed that with each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, and real numbers— the distributive law continues to hold, and the commutative and associative laws are still valid for both addition and multiplication. However, in Mathematics II students go further along this path. In standard N-RN.3, they explain that the sum or product of two rational numbers is rational, by say, arguing that the sum of two fractions with integer numerator and denominator is also a fraction of the same type, showing that the rational numbers are *closed* under the operations of addition and multiplication (MP.3). Moreover, they argue that the sum of a rational and irrational is irrational, and the product of a non-zero rational and an irrational is still irrational, showing that the irrational numbers are truly an additional set of numbers which along with the rational numbers forms a larger *system*, the real numbers (MP3, MP7).

With N-RN.1, students make meaning of the representation of radicals with rational exponents. Students are first introduced to exponents in grade six; by now, they should have an understanding of the basic properties of exponents (e.g. that $x^n \cdot x^m = x^{n+m}, (x^n)^m = x^{nm}, \frac{x^n}{x^m} = x^{n-m}, x^0 = 1 \text{ for } x \neq 0, \text{etc.}$). In fact, they may have justified certain properties of exponents by reasoning with others (MP.3, MP.7), for example, justifying why any nonzero number to the power 0 is equal to 1:

$$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1, \text{for } x \neq 0.$$

They further their understanding of exponents in Mathematics II by using these properties to explain the meaning of rational exponents. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be the same as $5^{[(1/3)\cdot 3]} = 5^1 = 5$ so that $5^{1/3}$ should represent the cube root of 5. In addition, the fact that $(ab)^n = a^n \cdot b^n$ reveals that

$$\sqrt{20} = (4 \cdot 5)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 2\sqrt{5}$$

showing that $\sqrt{20} = 2\sqrt{5}$. The intermediate steps of writing the square root as a rational exponent are not entirely necessary here, but the principle of how to work with radicals based on the properties of exponents is. Students extend such work with radicals and rational exponents to variable expressions as well (N.CN.2), e.g. rewriting an expression like $(a^2b^5)^{3/2}$ using radicals (N-RN.2).

The Complex Number System N-CN

Perform arithmetic operations with complex numbers. [i^2 as highest power of i]

1. Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form *a* + *bi* with *a* and *b* real.
2. Use the relation $i^2 = –1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Quadratics with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

In Mathematics II, students work with examples of quadratic functions and solve quadratic equations, where they encounter situations in which a resulting equation does not have a solution that is a real number, e.g. $(x-2)^2 = -25$. Here, students expand their extension of the concept of number to include complex numbers, numbers of the form $a + bi$, where i is a number with the property that $i^2 = -1$, so that such an equation can be solved. They begin to work with complex numbers, first by finding simple square roots of negative numbers: $\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25} \cdot \sqrt{-1} = 5i$ (MP.7). They also apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations like those above:

$$(x-2)^2 = -25, \text{ which implies } |x-2| = 5i, \text{ or } x = 2 \pm 5i.$$

Now equations like these have solutions, and the extended number system forms yet another system that behaves according to certain rules and properties (N-CN.1-2, N-CN.7-9). In addition, by exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show it is true for a quadratic polynomial by an application of the quadratic formula, and an understanding of the relationship between roots of a quadratic equation and the linear factors of the quadratic expression (MP.2).

Conceptual Category: Algebra

Students began their work with expressions and equations in the middle grades standards and extended their work to more complex expressions in Mathematics I. In Mathematics II, students encounter quadratic expressions for the first time and learn a whole new set of strategies for working with them. As in Mathematics I, the Algebra conceptual category is closely tied to the Functions conceptual category, linking the writing of equivalent expressions, solving equations, and graphing to function concepts.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Quadratic and exponential.]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems. [Quadratic and exponential.]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

In Mathematics II, students extend their work with expressions to include examples of more complicated expressions, such as those that involve multiple variables and exponents. Students use the distributive property to investigate equivalent forms of quadratic expressions, e.g. by writing $(x + y)(x - y) = x(x - y) +$

$y(x-y) = x^2 - xy + xy - y^2 = x^2 - y^2$, thereby yielding a special case of a factorable quadratic, the difference of squares. Students factor second- and simple third-degree polynomials by making use of such special forms, and by using factoring techniques based on properties of operations, e.g. factor-by-grouping, which arises from the distributive property (A-SSE.2). Note that the standards avoid talking about “simplification,” because it is often not clear what the simplest form of an expression is, and even in cases where it is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as the example below shows (Progressions, Algebra 2012, 4).

Example: *Which is the simpler form?* A particularly rich mathematical investigation involves finding a general expression for the sum of the first n consecutive natural numbers:

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

A famous tale speaks of a young C.F. Gauss being able to add the first 100 natural numbers quickly in his head, wowing his classmates and teacher alike. One way to find this sum is to consider the “reverse” of the sum:

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1,$$

and to then add the two expressions for S together, obtaining:

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1),$$

where there are n terms of the form $(n+1)$. Thus, $2S = n(n+1)$, so that $S = n(n+1)/2$. While students may be tempted to transform this expression into $\frac{1}{2}n^2 + \frac{1}{2}n$, they are obscuring the ease with which they can evaluate the first expression. Indeed, since n is a natural number, one of either n or $n+1$ is even, so evaluating $n(n+1)/2$, especially mentally, is often easier. Indeed, in Gauss’ case, $\frac{100(101)}{2} = 50(101) = 5050$.

Students also use different forms of the same expression to reveal important characteristics of the expression. For instance, when working with quadratics, they complete the square in the expression $x^2 - 3x + 4$ to obtain the equivalent expression $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$. Students can then reason with the new expression that the term being squared is greater than or equal to 0; hence, the value of the expression will always be greater than or equal to $\frac{7}{4}$ (A-SSE.3, MP.3). A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave, further contributing to students' understanding of work with expressions (MP.5).

Arithmetic with Polynomials and Rational Expressions **A-APR**

Perform arithmetic operations on polynomials. [Polynomials that simplify to quadratics]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

To perform operations with polynomials meaningfully, students are encouraged to draw parallels between the set of integers and the set of all polynomials with real coefficients (A-APR.1, MP.7). Understanding of the addition and subtraction of polynomials and the multiplication of monomials and binomials can be supported using manipulatives, such as "algebra tiles," which can offer a concrete representation of the terms in a polynomial (MP.5). The tile representation relies on the *area interpretation of multiplication*, the notion that the product ab can be thought of as the area of a rectangle of dimensions a units and b units. With this understanding, tiles can be used to represent 1 square unit (a 1×1 tile), x square units (a $1 \times x$ tile), and x^2 square units (an $x \times x$ tile). Finding the product $(x+5)(x+3)$ amounts to finding the area of an abstract rectangle of dimensions $x+5$ and $x+3$ as illustrated in the figure (MP.2).

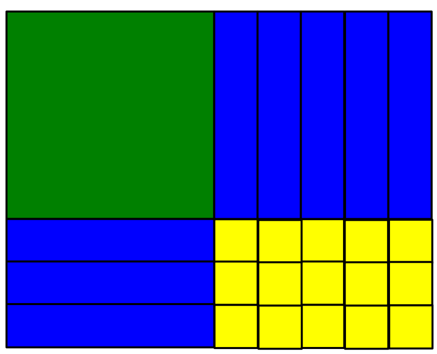


Figure 2: The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $\mathbf{x^2+5x+3x+15=x^2+8x+15}$.

Care must be taken in the way negative numbers are handled with this representation, since, as with all models, there are potential limitations to connecting the mathematics to the representation. The tile representation of polynomials can support student understanding of the meaning of multiplication of variable expressions, and is very useful for understanding the notion of completing the square, as described below.

Creating Equations **A-CED**

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA★**
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ [Include formulas involving quadratic terms.]

In Mathematics II, students work with all available types of functions to create equations, including quadratic functions, but constrained to simple cases (A-CDE.1). While functions used in A-CED.1, 2, and 4 will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I. Note that students are not required to study rational functions in Mathematics II.

Reasoning with Equations and Inequalities **A-REI**

Solve equations and inequalities in one variable. [Quadratics with real coefficients]

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations. [Linear-quadratic systems.]

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Students in Mathematics II extend their work with exponents to working with quadratic functions and equations. To extend their understanding of these quadratic expressions and the functions they define, students investigate properties of quadratics and their graphs in the Functions domain. The solving of quadratic equations can perhaps be best presented in the context of functions. For instance, if the equation $h(t) = -16t^2 + 50t + 150$ defines the height of a projectile launched with an initial velocity of 50 ft/s from a height of 150 ft, then asking at which time t the object hits the ground is asking for which t we have $h(t) = 0$. That is, we now need to solve the equation $-16t^2 + 50t + 150 = 0$ and need new methods for doing so. Students have investigated how to "undo" linear and simple exponential functions in Mathematics I; now they do so for quadratic functions and discover the process is more complex.

Example: When solving quadratic equations of the form $(x-p)^2 = q$ in standard A-REI.4.a, students rely on the understanding that they can take square roots of both sides of the equation to obtain,

$$\sqrt{(x-p)^2} = \sqrt{q}. \qquad (1)$$

In the case that $\sqrt{q}$ is a real number, we can solve this equation for x. A common mistake is to quickly introduce the symbol ± here, without understanding where it comes from. Doing so without care often leads to students thinking that $\sqrt{9} = \pm 3$, for example.

Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = |a|$. Applying this to equation (1) yields $|x-p| = \sqrt{q}$. Solving this simple absolute value equation yields that $x - p = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the solutions $x = p + \sqrt{q}, p - \sqrt{q}$.

Students transform quadratic equations into the form $ax^2 + bx + c = 0$, for $a \neq 0$, the *standard form* of a quadratic equation. In some cases, the quadratic expression factors nicely and students can apply the *zero product property of the real numbers* to solve the resulting equation. The zero product property states that for two real numbers m and n, $m \cdot n = 0$ if and only if either $m = 0$ or $n = 0$. Hence, when a quadratic polynomial can be factored and rewritten as $a(x-r)(x-s) = 0$, the solutions can be found by setting each of the linear factors equal to 0 separately, and obtaining the solution set $\{r, s\}$. In other cases, a means for solving a quadratic equation arises by *completing the square*. Assuming for simplicity that $a = 1$ in the standard equation above, and the equation has been rewritten as $x^2 + bx = -c$, we can "complete the square" by adding the square of half the coefficient of the x-term to each side of the equation:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2. \qquad (2)$$

The result of this simple step is that the quadratic on the left side of the equation is a perfect square,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

Thus, we have now converted equation (2) into an equation of the form $(x - p)^2 = q$, namely,

$$\left(x + \frac{b}{2}\right)^2 = -c + \frac{b^2}{4},$$

which can be solved by the method described above, as long as the term on the right is positive. The case when $a \neq 1$ can be handled similarly and ultimately results in the familiar quadratic formula. Tile representations of quadratics illustrate that the process of completing the square has a geometric interpretation that explains the origin of the name. Students should be encouraged to explore these representations in order to make sense out of the process of completing the square (MP.1, MP.5).

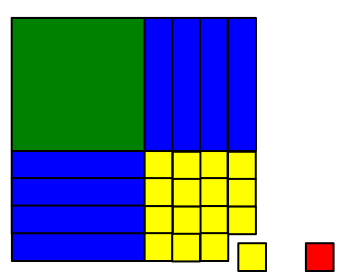
Completing the Square: The method of completing the square is a useful skill in Algebra. It is generally used to change a quadratic in standard form, $ax^2 + bx + c$, into one in vertex-form, $a(x - h)^2 + k$. The vertex form can help determine several properties of quadratic functions. Completing the square also has applications in Geometry (G-GPE.1) and later higher math courses.

Example: To complete the square for the quadratic $y = x^2 + 8x + 15$, we take half the coefficient of the x-term and square it to yield 16. We realize that we need only to add 1 and subtract 1 to the quadratic expression:

$$y = x^2 + 8x + 15 + 1 - 1 = x^2 + 8x + 16 - 1.$$

Factoring gives us $y = (x + 4)^2 - 1$.

In the picture, note that the tiles used to represent $x^2 + 8x + 15$ have been rearranged to try to form a square, and that a positive unit tile and a negative unit tile are added into the picture to "complete the square."



Conceptual Category: Geometry

Students began to formalize their understanding of geometry in Mathematics I, by defining congruence in terms of well-defined rigid motions of the plane. They found that congruence could be deduced in certain cases by investigating other relationships, e.g. that for triangles the ASA, SAS, and SSS congruence criteria held. In Mathematics II, students further enrich their ability to reason deductively and begin to write more formal proofs of various geometric results. In addition, they apply their knowledge of similarity to triangles and discover powerful relationships in right triangles, leading to the discovery of trigonometric functions. Finally, the Pythagorean relationship and students' work with quadratics leads to algebraic representations of circles and more complex proofs of results in the plane.

Geometry

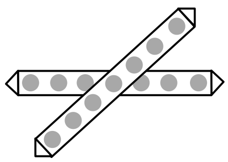
Congruence **G-CO**

Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Students prove the congruence criteria (SSS, SAS, ASA) for triangles using the more basic notion of congruence by rigid motions. Instructors are encouraged to use a variety of strategies for engaging students in understanding and writing proofs, including: using ample pictures to demonstrate results; using patty paper, transparencies, or dynamic geometry software to explore the relationships in a proof; creating flow charts and other organizational diagrams for outlining a proof; and writing step-by-step or paragraph formats for the completed proof (MP.5). Above all else, the reasoning involved in connecting one step in the logical argument to the next should be emphasized. Students should be encouraged to make conjectures based on experimentation, to justify their conjectures, and to communicate their reasoning to their peers (MP.3). Such reasoning, justification, and communication in precise language are at the heart of any geometry instruction, and should be focused on here.

Example. *The Kite Factory*. The hypothetical situation of a kite factory is presented to students, wherein kite engineers wish to know how the shape of a kite affects how it flies (e.g. the lengths of the rods, where they are attached, the angle at which



they are attached, etc.). In this activity, students are given pieces of cardstock of various lengths, hole-punched at regular intervals so they can be attached in different places.

These two "rods" form the frame for a kite at the kite factory. By changing the angle at which the sticks are held, and the places where they are attached, students discover different properties of quadrilaterals.

Students are challenged to make conjectures and use precise language to describe their findings about which diagonals result in which quadrilaterals. They can discover properties unique to certain quadrilaterals, such as the fact that diagonals that are perpendicular bisectors of each other imply the quadrilateral is a rhombus. See videos of this lesson being implemented in a high school classroom at *insidemathematics.org*.

Similarity, Right Triangles, and Trigonometry — G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

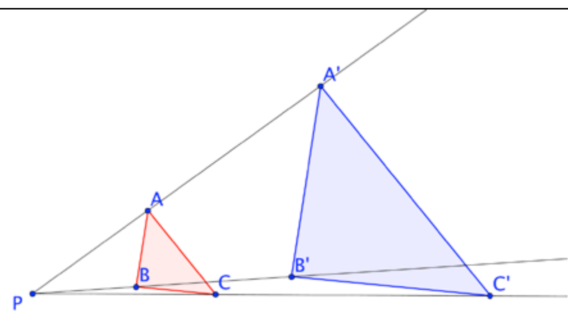
Prove theorems involving similarity. [Focus on validity of underlying reasoning while using variety of formats.]

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

As right triangles and triangle relationships play such an important role in applications and future mathematics learning, they are given a prominent role in the geometry conceptual category. A discussion of similarity is necessary first, and again, a more precise mathematical definition is given in the higher mathematics curriculum. Students have worked with *dilations* as a transformation in the grade eight standards; now, they explore the properties of dilations in more detail and develop an understanding of the notion of *scale factor* (G-SRT.1). Whereas previously objects that were similar simply had "the same shape," the definition taken for two objects being similar is that there is a sequence of *similarity* transformations[2] of the plane that maps one exactly onto the other. In standards G-SRT.2 and G-SRT.3, students explore the consequences of two triangles being similar: that they have congruent angles and that their side lengths are in the same proportion. This new understanding gives rise to more results, encapsulated in standards G-SRT.4 and G-SRT.5

Example: Experimenting with dilations.

Students are given opportunities to experiment with dilations and determine how they affect planar objects. Students first make sense of the definition of a *dilation of scale factor* $k > 0$ *with center* P as the transformation that moves a point A along the ray PA to a new point A', so that $|PA'| = k \cdot |PA|$. For example, students apply the dilation of scale factor 2.5 with center P to the points A, B, and C illustrated using a ruler. Once they've done so, they consider the two triangles ΔABC and $\Delta A'B'C'$. What they discover is that the lengths of the corresponding sides of the triangles have the same ratio as dictated by the scale factor. (G-SRT.2) Students learn that parallel lines are taken to parallel lines by dilations; thus corresponding



segments of ΔABC and $\Delta A'B'C'$ are parallel. After students have proved results about parallel lines intersected by a transversal, they can deduce that the angles of the triangles are congruent. Through experimentation, they see that congruent corresponding angles is a necessary and sufficient condition for the triangles to be similar, leading to the *AA* criterion for triangle similarity. (G.SRT.3.)

Similarity, Right Triangles, and Trigonometry G-SRT

[2] Translation, rotation, reflection, or dilation.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90°and 45°, 45°, 90°). CA

Once the angle-angle (AA) similarity criterion for triangles is established, it follows that any two *right* triangles ΔABC and ΔDEF with at least one pair of angles congruent (say $\angle A \cong \angle D$) is similar, since the right angles are obviously congruent (say $\angle B \cong \angle E$). By similarity, the corresponding sides of the triangles are in proportion:

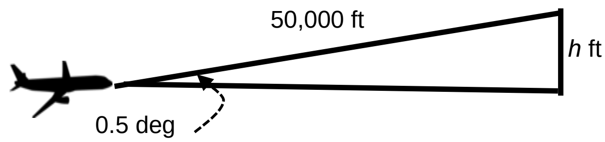
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Notice the first and third expressions in the statement of equality above can be rearranged to yield that

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

Since the triangles in question are arbitrary, this implies that for any right triangle with an angle congruent to $\angle A$, the ratio of the side adjacent to $\angle A$ and the hypotenuse of the triangle is a certain constant. This allows us to unambiguously define the *sine* of $\angle A$, denoted by $\sin A$, as this ratio. In this way, students come to understand the trigonometric functions as relationships completely determined by angles (G-SRT.6). They further their understanding of these functions by investigating relationships between sine, cosine, and tangent, by exploring the relationship between the sine and cosine of complementary angles, and by applying their knowledge of right triangles to real-world situations, (MP.4) such as in the example below (G-SRT.6-8). Experience working with many different triangles, finding their measurements, and computing ratios of the measurements found will help students understand the basics of the trigonometric functions.

Example: *Using Trigonometric Relationships.* Planes that fly at high speed and low elevations	*Solution:* The sketch of the situation below shows that there is a right triangle with hypotenuse 50,000

often have onboard radar systems to detect possible obstacles in the path of the plane. The radar can determine the range of an obstacle and the angle of elevation to the top of the obstacle. Suppose that the radar detects a tower that is 50,000 feet away, with an angle of elevation of 0.5 degree. By how many feet must the plane rise in order to pass above the tower?	(ft) and smallest angle 0.5 (degree). To find the side opposite this angle, which represents the minimum height the plane should rise, we use $\sin 0.5 = \frac{h}{50{,}000}$, so that $h = (50{,}000)\sin 0.5 \approx 436.33$ ft. 

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

Students can extend their understanding of the usefulness of similarity transformations through investigating circles (G-C.1). For instance, students can reason that any two circles are similar by describing precisely how to transform one into the other.

Example. Students can show that the two circles C and D given by the equations below are similar. $C: (x-1)^2 + (y-4)^2 = 9$ $D: (x+2)^2 + (y-1)^2 = 25$ **Solution.** Since the centers of the circles are $(1, 4)$ and $(-2, 1)$, respectively, we first translate the	center of circle C to the center of circle D using the translation $T(x, y) = (x-3, y-3)$. Finally, since the radius of circle C is 3 and the radius of circle D is 5, we dilate from the point $(-2, 1)$ by a scale factor of $5/3$.

Students continue investigating properties of circles, and relationships among angles, radii and chords (G-C.2, 3, 4).

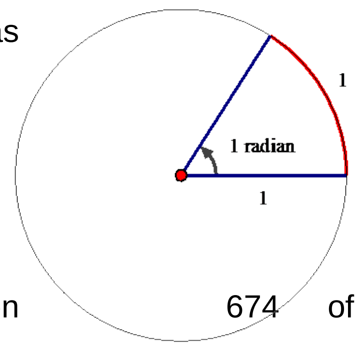
Another interesting application of the notion of similarity is the definition of the radian measure of an angle. Students can derive this result in the following way: given a sector of a circle C of radius r and central angle α, and a sector of a circle D of radius s and central angle also α, it follows that

$$\frac{\text{length of arc on circle } C}{r} = \frac{\text{length of arc on circle } D}{s}.$$

Therefore, much like when defining the trigonometric functions, there is a constant m such that for an arc subtended by an angle α on any circle, we have

$$\frac{\text{length of arc subtended by angle } \alpha}{\text{radius of the circle}} = m.$$

This constant of proportionality is the *radian measure* of angle α. It follows that an angle that subtends an arc on a circle that is the same length as the radius measures 1 radian. By investigating circles of different sizes, measuring off arcs subtended by the same angle using string, and finding the ratios described above, students can apply their proportional reasoning skills to discover this constant ratio, thereby understanding the definition of radian measure.



Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$*) lies on the circle centered at the origin and containing the point (0, 2).* [Include simple circle theorems.]
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

The largest intersection of traditional algebraic and geometric concepts occurs here, wherein two-dimensional shapes are represented on a coordinate system and can be described using algebraic equations and inequalities. Readers will be familiar with the derivation of the equation of a circle by the Pythagorean Theorem and the definition of a circle (G-GPE.1). Given that a circle consists of all points (x, y) that are at a distance $r > 0$ from a fixed center (h, k), we see that $\sqrt{(x-h)^2 + (y-k)^2} = r$ for any point lying on the circle, so that $(x-h)^2 + (y-k)^2 = r^2$ determines this circle. Students can derive this equation, and flexibly change an equation into this form by completing the square as necessary. By understanding the derivation of this equation, the variables h, k and r take on a clear meaning. Students do the same for the definition of a parabola in terms of a focus and directrix in G-GPE.2. Ample resources are available for application problems involving parabolas, and should be explored to connect the geometric and algebraic aspects of these curves.

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★
5. **Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k², and k³, respectively; determine length, area and volume measures using scale factors. CA★**
6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

The ability to visualize two- and three-dimensional shapes is useful skill. This group of standards addresses that skill, and includes understanding and using volume and area formulas for curved objects, which students have not yet been responsible for. Students also have the opportunity to make use of the notion of a *limiting process*, an idea that plays a large role in calculus and advanced mathematics courses, when they investigate the formula for the area of a circle. By experimenting with grids of finer and finer mesh for example, they can repeatedly approximate the area of a unit circle, and thereby get a better and better approximation for the irrational number π. They also dissect shapes and make arguments based on the dissections. For instance, a cube can be dissected into three congruent pyramids, as shown in the figure, which can lend weight to the formula that the volume of a pyramid of base area B and height h is $\frac{1}{3}Bh$ (MP.2).

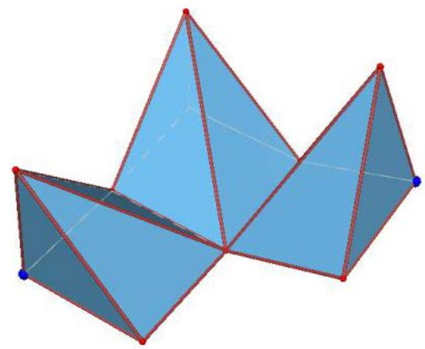


Figure 3: Three congruent pyramids that make a cube.

(Park City Mathematics Institute 2013)

Conceptual Category: Statistics and Probability

In grades seven and eight, students learned some basics of probability, including chance processes, probability models, and sample spaces. In high school, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value (Progressions, Statistics and Probability 2011). Building on probability concepts that began in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions (CCSSI 2010).

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*), and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, *P*(*A* and *B*) = *P*(*A*)*P*(*B*|*A*) = *P*(*B*)*P*(*A*|*B*), and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

To develop student understanding of conditional probability, students should experience two types of problems: ones in which the uniform probabilities attached to outcomes lead to independence of the outcomes, and ones in which they do not (S-CP.1-3). Below are two examples wherein these two possibilities occur.

Example (Adapted from Progressions, Statistics and Probability 2011). *Guessing On a True-False Quiz*. If there are four T-F questions on a quiz, then the possible outcomes based on guessing on each question can be arranged as in the table below:	By simple counting outcomes one can find various probabilities. For example, $$P(\text{C on first question}) = \frac{1}{2},$$ and $$P(\text{C on second question}) = \frac{1}{2},$$

Possible outcomes: Guessing on four true–false questions

Number correct	Outcomes	Number correct	Outcomes	Number correct	Outcomes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

as well. Noticing that

$$P[(\text{C on first})\text{ AND }(\text{C on second})]=\frac{4}{16}=\frac{1}{2}\cdot\frac{1}{2},$$

shows that the two events, getting the first question correct and the second correct, are independent.

Example (Adapted from Progressions, Statistics and Probability 2011). *Working-Group Leaders.* Suppose a 5-person working group consisting of three girls (*A*pril, *B*riana, and *C*yndi) and two boys (*D*aniel and *E*rnesto) wants to randomly choose two people to lead the group. The first person is the discussion leader and the second is the recorder, so order is important in selecting the leadership team. There are 20 outcomes for this situation, shown below:

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Notice that the probability of selecting two girls as the leaders is:

$$P(\text{two girls chosen})=\frac{6}{20}=\frac{3}{10},$$

whereas

$$P(\text{girl selected on first draw})=\frac{12}{20}=\frac{3}{5},$$

while

$$P(\text{girl selected on second draw})=\frac{3}{5},$$

as well. But since $\frac{3}{5}\cdot\frac{3}{5}\neq\frac{3}{10}$, the two events are not independent.

One can also use the conditional probability perspective to show these events are not independent. Since

$$P(\text{girl on second}\mid\text{girl on first})=\frac{6}{12}=\frac{1}{2},$$

and $P(\text{girl on second})=\frac{3}{5}$, these events are seen to be dependent.

Students also explore finding probabilities of compound events (S-CD.6-9) by using the Addition Rule (that $P(A\text{ OR }B)=P(A)+P(B)-P(A\text{ AND }B)$, and the general Multiplication Rule $P(A\text{ AND }B)=P(A)\cdot P(B|A)=P(B)\cdot P(A|B)$. A simple experiment involving rolling two dice and tabulating the possibly outcomes can shed light on these formulas before they are extended to application problems.

Example (Adapted from Illustrative Mathematics 2013). On April 15, 1912, the class passengers who survived. That is, we restrict the sample space to only first class passengers to

Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. Students can use the fact that two events A and B are independent if $P(A\|B) = P(A) \cdot P(B)$. Let A represent the event a passenger survived, and B represent the event that the passenger was in first class. We compare the conditional probability $P(A\|B)$ with the probability $P(A)$. The probability of surviving, given that the passenger was in first class, is the fraction of first	obtain $P(A\|B) = \frac{202}{325} \approx 0.622$. The probability that the passenger survived is the number of all passengers who survived divided by the total number of passengers, that is $P(A) = \frac{498}{1316} \approx 0.378$. Since $0.622 \neq 0.378$, the two given events are not independent. Moreover, we can say that being a passenger in first class increased the chances of surviving. Students can be challenged to further investigate where similar reasoning would apply today. For example, what are similar statistics for Hurricane Katrina, and what would a similar analysis conclude about the distribution of damages? (MP.4)

Using Probability to Make Decisions **S-MD**

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Standards S-MD.6 and S-MD.7 involve students' use of probability models and probability experiments to make decisions. These standards set the stage for more advanced work in Mathematics III, i.e. where the ideas of statistical inference are introduced. See the "High School Progression on Statistics and Probability" for more explanation and examples: http://ime.math.arizona.edu/progressions/.

Mathematics II Overview

Number and Quantity

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

The Complex Number Systems

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve equations and inequalities in one variable.
- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic and exponential models and solve problems.
- Interpret expressions for functions in terms of the situations they model.

Trigonometric Functions

- Prove and apply trigonometric identities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Congruence

- Prove geometric theorems.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.

Statistics and Probability

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions

★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making
(+) Indicates additional mathematics to prepare students for advanced courses

Mathematics II

Number and Quantity

The Real Number System N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

The Complex Number System N-CN

Perform arithmetic operations with complex numbers. [i^2 as highest power of i]

1. Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Quadratics with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Quadratic and exponential]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Polynomials that simplify to quadratics]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Creating Equations A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA**★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ [Include formulas involving quadratic terms.]

Reasoning with Equations and Inequalities A-REI

Solve equations and inequalities in one variable. [Quadratics with real coefficients]

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations. [Linear-quadratic systems.]

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = –3x$ *and the circle* $x^2 + y^2 = 3$.

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.*
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities. [Quadratic and exponential]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. ★

Build new functions from existing functions. [Quadratic, absolute value]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Include quadratic.]

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

6. Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA★

Trigonometric Functions F-TF

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Geometry

Congruence G-CO

Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity. [Focus on validity of underlying reasoning while using variety of formats.]

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90°and 45°, 45°, 90°). CA

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$ *) lies on the circle centered at the origin and containing the point (0, 2).* [Include simple circle theorems.]
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★
5. **Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area and volume measures using scale factors. CA★**
6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

Statistics and Probability

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*), and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, *P*(*A* and *B*) = *P*(*A*)*P*(*B*|*A*) = *P*(*B*)*P*(*A*|*B*), and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★