Mathematics I

Introduction

The following Integrated and Traditional Pathways lay out two sequences of courses that include the “College and Career Readiness” standards of the higher mathematics CA CCSSM. These are the standards deemed appropriate for all students to learn in preparation for pursuing a college education or a career in a technology-infused 21st Century. Most of the additional “plus” (+) standards are treated in courses beyond these pathways and are intended to be learned by those students who wish to pursue higher-level mathematics courses or college majors in a STEM field.

These pathways and the two suggested “fourth courses” presented in the framework do not represent the only courses available for high school students. The higher mathematics CA CCSSM are designed in part as a menu of standards from which educators can create customized courses. Thus, for example, a school or district may opt to create a mathematics course based on certain CA CCSSM and the Career Technical Education Model Curriculum Standards, or they may opt to create a Mathematical Modeling course using starred (★) standards (see the Appendix, “Mathematical Modeling,” for more information). Overall, schools and districts have a choice in their course offerings at the high school level and are welcome to take advantage of the focus on real-world applications of the higher mathematics standards and the overall flexibility of the CA CCSSM.

The fundamental purpose of Mathematics I is to formalize and extend students’ understanding of linear functions and their applications. The critical topics of study deepen and extend understanding of linear relationships, in part by contrasting them with exponential phenomena, and in part by applying linear models to data that exhibit a linear trend. Mathematics I uses properties and theorems involving congruent figures to deepen and extend understanding of geometric knowledge from prior grades. The courses in the Integrated Pathway follow the structure began in the K-8 standards of presenting mathematics as a coherent subject, mixing standards from various conceptual categories.

The standards in the integrated Mathematics I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course will be expanded upon below according to these conceptual categories, but teachers and administrators alike should note that the standards are not topics to be checked off a list during isolated units of instruction, but rather content that should be present throughout the school year through rich instructional experiences. In addition, the standards should not necessarily be taught in the order in which they appear here, but rather in a coherent manner.

What Students learn in Mathematics I

Overview

In Mathematics I, students continue their work with expressions and modeling and analyzing situations. In earlier grades, students informally define, evaluate, and compare functions, and use them to model relationships between quantities. In Mathematics I, students will learn function notation and develop the concepts of domain and range. They move beyond viewing functions as processes that take inputs and yield outputs and start viewing functions as objects that can be combined with operations (e.g., finding $(f+g)(x) = f(x) + g(x)$.). They explore many examples of functions, including sequences. They interpret functions represented graphically, numerically, symbolically, and verbally, translate between representations, and understand the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that, depending upon the context these representations are likely to be approximate and incomplete. Their work includes functions that can be described or approximated by formulas as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

By the end of the eighth grade content standards, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Mathematics I builds on these earlier experiences by asking students to analyze and explain the process of solving an equation and to justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating between various forms of linear equations and inequalities, and using them to solve problems. They master solving linear equations and apply related solution techniques and the laws of exponents to the creation and solving of simple exponential equations. Students explore systems of equations and inequalities, and they find and interpret their solutions. All of this work is grounded on understanding quantities and on relationships between them.

In Mathematics I, students build on their prior experiences with data, developing more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

In previous grades, students were asked to draw triangles based on given measurements. They also have prior experience with rigid motions: translations, reflections, and rotations, and have used these to develop notions about what it means for two objects to be congruent. In Mathematics I, students establish triangle congruence criteria, based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work. Finally, building on their work with the Pythagorean Theorem in the grade eight standards to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

Examples of Key Advances from Grades K-8

- Students build on previous work with solving linear equations and systems of linear equations from seventh and eighth grade in two ways: (a) They extend to more formal solution methods, including attending to the structure of linear expressions, and (b) they solve linear inequalities.
- Students formalize their understanding of the definition of a function, particularly their understanding of linear functions, emphasizing the structure of linear expressions. Students also begin to work with exponential functions, comparing them to linear functions.
- Work with congruence and similarity transformations that was begun in grades 6-8 progresses. Students consider sufficient conditions for the congruence of triangles.
- Work with bivariate data and scatter plots in grades 6-8 is extended to working with lines of best fit. [From the PARCC for HS, 26]

Connecting Standards for Mathematical Practice and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics* in the classroom and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the MP standards. Connecting content and practices happens in the context of *working on problems*; the very first MP standard is to make sense of problems and persevere in solving them. The table below gives examples of how students can engage in the MP standards in Mathematics I.

Standards for Mathematical Practice *Students…*	**Examples of each practice in Mathematics I**
MP1. Make sense of problems and persevere in	Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric

solving them.	problems to help them make sense of the problems.
MP2. Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students reason through the solving of equations, recognizing that solving an equation is more than simply a matter of rote rules and steps. They use language such as "if… then..." when explaining their solution methods
MP4. Model with mathematics.	Students apply their mathematical understanding of linear and exponential functions to many real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and examining patterns in data from real-world contexts.
MP5. Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.
MP6. Attend to precision.	Students use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem.
MP7. Look for and make use of structure.	Students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.
MP8. Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2-y_1}{x_2-x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y-y_1}{x-x_1}$ will give a general equation of that line.

MP standard 4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Though the Modeling category has no specific standards listed within it, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Readers will see some standards marked with a star symbol (★) to indicate that they are *modeling standards*, that is, they present an opportunity for applications to real-world modeling situations more so than other standards. Modeling with mathematics is a theme in all higher math courses. Modeling problems in higher mathematics center on problems arising in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the current course.

Examples of places where specific MP standards can be implemented in the Mathematics I standards will be noted in parentheses, with the specific practice standard(s) indicated.

Mathematics I Content Standards by Conceptual Category

The Mathematics I course is organized by conceptual category, domains, clusters, and then standards. Below, the overall purpose and progression of the standards included in Mathematics I are described according to these conceptual categories. Note that the standards are not listed in an order in which they should be taught. Standards that are considered to be new to secondary grades teachers will be discussed in more depth than others.

Conceptual Category: Modeling

Throughout the higher mathematics CA CCSSM, certain standards are marked with a (⋆) symbol to indicate that they are considered modeling standards. Modeling at this level goes beyond the simple application of previously constructed mathematics to real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: which of the quantities present in this situation are known and unknown? Students need to decide on a solution path; sometimes that path may need to be revised. They will make use of tools such as calculators, dynamic geometry software, or spreadsheets. They will try to use previously derived models (e.g. linear functions) but may find that a new formula or function will apply. They may see that solving an equation arises as a necessity when trying to answer their question, and that oftentimes the equation arises as the specific instance of the knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. This will be a new approach for many teachers and will be challenging to implement, but the effort will produce students who can appreciate that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

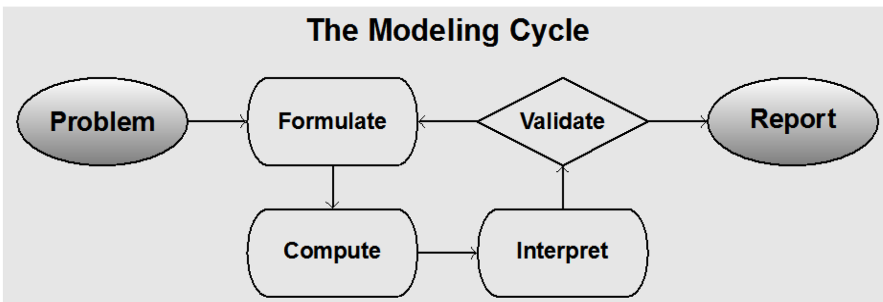


Figure 1: The modeling cycle. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret their results, validate them, and report out.

Throughout the Mathematics I chapter, the examples given will be framed as much as possible as modeling situations, to serve as illustrations of the concept of mathematical modeling. The big ideas of linear and exponential functions, graphing, solving equations, and change will be explored through this lens. The reader is encouraged to consult the Appendix, “Mathematical Modeling,” for a further discussion of the Modeling Cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can serve as motivation for studying the standards in the other Mathematics I conceptual categories. For instance, an equation wherein one is asked to “solve for x” can be seen as a search for the input of a function f that gives a specified output, and solving the equation amounts to undoing the work of the function. Or, the graph of an equation such as $y = \frac{1}{3}x + 5$, can be seen as a certain representation of a function f where $f(x) = \frac{1}{3}x + 5$. Solving a more complicated equation can be seen as asking for which values of x do two functions f and g agree (i.e. when does $f(x) = g(x)$?) and the intersection of the two graphs $y = f(x)$ and $y = g(x)$ is then connected to the solution of this equation. In general, functions describe in a precise way how two quantities are related, and can be used to make predictions and generalizations, keeping true to the emphasis on modeling in higher mathematics.

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often assume the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. When describing relationships between quantities, the defining characteristic of a *function* is that the input value determines the output value, or equivalently, that the output value depends upon the input value (The University of Arizona Progressions Documents for the Common Core Math Standards [Progressions], Functions 2012, 2).

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an assignment, such as the fact that each individual is given a unique Social Security Number; by an algebraic expression like $f(x) = a + bx$; or by a recursive rule, such as $f(n+1) = f(n) + b, f(0) = a$. The graph of a function is often a useful way of visualizing the relationship that the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Interpreting Functions **F-IF**

Understand the concept of a function and use function notation. [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then *f*(*x*) denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation *y* = *f*(*x*).
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.).
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by f(0) = f(1) = 1, f(n + 1) = f(n) + f(n − 1) for n ≥ 1.*).

Interpret functions that arise in applications in terms of the context. [Linear and exponential (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

While the grade eight standards called for students to work informally with functions, in Mathematics I they begin to refine their understanding and use the formal mathematical language of functions. Standards F-IF.1-9 deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (MP.6). In F-IF.3, students learn the language of functions and that a function has a domain that should be specified. For instance, the equation $f(x) = x^2$ by itself does not describe a function entirely. Similarly, though the expressions in the equations $f(x) = 3x - 4$ and $g(n) = 3n - 4$ look the same except for the variables used, f may have as its domain all real numbers, while g may have as its domain the natural numbers (i.e. g defines a sequence). Students make the connection between the graph of the equation $y = f(x)$ and the function itself—namely, that the coordinates of any point on the graph represent an input and output, expressed as $(x, f(x))$—and understand that the graph is a *representation* of a function. They connect the domain and range of a function to the graph (F-IF.5). Note that there is neither an exploration of the notion of *relation vs. function* nor the *vertical line test* in the CA CCSSM. This is by design. The core question when investigating functions is: “Does each element of the domain correspond to exactly one element of the range?” (Progressions, Functions 2012, 8)

Standard F-IF.3 introduces sequences as functions. In general, a sequence is a function whose inputs consist of a subset of the integers, such as: $\{0,1,2,3,4,5,\dots\}$. Students can begin to study sequences in simple contexts, such as when calculating their total pay P when working for n days at $\$65$ per day, obtaining a general expression $P(n) = 65 \cdot n$. Students investigate geometric sequences of the form $g(n) = ar^n$, $n \geq 1$, or $g(1) = ar, g(n+1) = r \cdot g(n),$ for $n \geq 2$, when they study population growth or decay, as in the availability of a medical drug over time, or financial mathematics, such as when determining compound interest. Notice that the domain is included in the description of the rule (Adapted from Progressions, Functions 2012, 8).

Interpreting Functions **F-IF**

Analyze functions using different representations. [Linear and exponential]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

In standards F-IF.7 and F-IF.9, students represent functions with graphs and identify key features in the graph. In Mathematics I students study only linear, exponential and absolute value functions. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context.

Building Functions **F-BF**

Build a function that models a relationship between two quantities. [For F.BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Knowledge of functions and expressions is only part of the picture. One must be able to understand a given situation and apply function reasoning to model how quantities change together. Often, the function created sheds light on the situation at hand; one can make predictions of future changes, for example. This is the content of the standards F-BF.1 and F-BF.2 (starred to indicate they are Modeling standards). Mathematics I features the introduction of arithmetic and geometric sequences, written both explicitly and recursively. Very often, students can see the recursive pattern of a sequence, that is, how the sequence changes from term to term, but they have a difficult time finding an explicit formula for the sequence.

For example, a population of cyanobacteria can double every 6 hours under ideal conditions, at least until the nutrients in its supporting culture are depleted. This means a population of 500 such bacteria would grow to 1000 in the first 6-hour period, 2000 in the second 6-hour period, 4000 in the third 6-hour period, etc. Evidently, if n represents the number of 6-hour periods from the start, the population at that time $P(n)$ satisfies $P(n) = 2 \cdot P(n-1)$. This is a *recursive* formula for the sequence $P(n)$, which gives the population at a given time period n in terms of the population at time period $n-1$. To find a closed, *explicit*, formula for $P(n)$, students can reason that

$$P(0) = 500, P(1) = 2 \cdot 500, P(2) = 2 \cdot 2 \cdot 500, P(3) = 2 \cdot 2 \cdot 2 \cdot 500, \ldots$$

A pattern emerges, that $P(n) = 2^n \cdot 500$. In general, if an initial population P_0 grows by a factor $r > 1$ over a fixed time period, then the population after n time periods is given by $P(n) = P_0 r^n$.

The following example shows how students can create functions based on prototypical ones such as this one.

Exponential Growth. The following example illustrates the type of problem that students can solve after they have worked with basic exponential functions like the one described above. **Example (Adapted from Illustrative Mathematics 2013).** On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.	**Some possible questions:** a. When will the lake be covered halfway? b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake. **Solution and Comment.** a. Since the population doubles each day, and since the entire lake is covered by June 30, this implies that half the lake was covered on June 29. b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, we can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = (1.86 \times 10^{-7})2^t$.

It should be noted that sequences often *do not* lend themselves to compact, explicit formulas like in the example above. Students will find this given enough examples. The means for deciding which do have explicit formulas, such as arithmetic and geometric sequences, is an important area of instruction.

The content of standard F-BF.3 has typically been left to later courses. In Mathematics I, the focus is on linear and exponential; even and odd functions are not addressed. In keeping with the theme of the input-output interpretation of a function, students should work towards developing an understanding of the effect on the output of a function under certain transformations, such as in the table below:

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x

Such understandings can help in seeing the effect of transformations on the graph of a function, and in particular, can aide in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable (e.g. the graph of $y = f(x+2)$ is the graph of f shifted 2 units to the left, not to the right) (Progressions, Functions 2012, 7).

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Interpret expressions for functions in terms of the situation they model. [Linear and exponential of form $f(x) = b^x + k$]

5. Interpret the parameters in a linear or exponential function in terms of a context.★

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. In standards F-LE.1a-c, students recognize and understand the defining characteristics of linear and exponential functions. Students have already worked extensively with linear equations, and have developed an understanding that an equation in two variables of the form $y = mx + b$ exhibits a special relationship between the variables x and y, namely, that a change of Δx in the variable x, the independent variable, results in a change of $\Delta y = m \cdot \Delta x$ in the dependent variable y. They have seen this informally, in graphs and tables of linear relationships, starting in the grade eight standards (8.EE.5, 8.EE.6, 8.F.3). If one considers only integer values of x, so that the incremental change in x is simply 1 unit, then the change in y is exactly m; it is this constant rate of change m that defines linear relationships, both in discrete linear sequences and in general linear functions of one real variable. Put differently, students recognize that for successive whole number input values, x and $x + 1$, a linear function $f(x) = mx + b$ exhibits a constant rate of change:

$$f(x+1) - f(x) = [m(x+1) + b] - (mx + b) = m(x + 1 - x) = m.$$

In contrast, exponential equations $g(n) = ab^n$, exhibit a constant *percent* change. For instance, a t-table for the equation $y = 3^n$, illustrates the constant ratio of successive y-values for this equation.

x	$y = 3^n$	Ratio of successive y-values
1	3	
2	9	9/3=3
3	27	27/9=3
4	81	81/27=3

This table shows that each value of *y* is 3 times the value preceding it (i.e. 300% of the value preceding it), illustrating the constant percent change of this exponential. In the general case, we have

$$\frac{g(n+1)}{g(n)} = \frac{ab^{n+1}}{ab^n} = \frac{b^{n+1}}{b^n} = b^{(n+1)-n} = b,$$

illustrating the constant ratio of successive values of g.[1]

The standards require students to prove the result above for linear functions (F-LE.1a). In general, students must also be able to recognize situations that represent both linear and exponential functions and construct functions to describe the situations (F-LE.2). Finally, students interpret the parameters in linear and exponential functions, and model physical problems with such functions.

A graphing utility, computer algebra system, or common spreadsheet can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions (MP.4). A real-world example where this can be explored involves investments, mortgages, and other financial instruments. Students can develop formulas for annual compound interest based on a general formula, such as $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$, where P_0 is the initial amount invested, r is the interest rate, n is the number of times the interest is compounded per year, and t is the number of years the money is invested. They can explore values after different time periods and compare different rates and plans using computer algebra software or simple spreadsheets (MP.5). This hands-on experimentation with such functions helps develop an understanding of their behavior.

Conceptual Category: Number and Quantity

In real-world problems, the answers are usually not numbers but *quantities*: numbers with units, which involves measurement. In their work in measurement up through grade eight, students primarily measure commonly used attributes such as length, area, and volume. In higher mathematics, students encounter a wider variety of units in modeling, e.g., when considering acceleration, currency conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

[1] In CA CCSSM Mathematics I, only integer values for x are considered in exponential equations such as $y = b^x$.

Quantities N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

In Mathematics I, students reason through problems with careful selection of units, and they use units to understand problems and make sense of the answers they deduce. Standards N-Q.1–3 are modeling standards that refer to students' appropriate use of units and definition of quantities. For instance, students can evaluate the accuracy of the following conclusion made in a magazine, "On average the human body is more than 50 percent water [by weight]. Runners and other endurance athletes average around 60 percent. This equals about 120 soda cans' worth of water in a 160-pound runner!" (Illustrative Mathematics 2013) They seek out appropriate unit conversions, such as that a typical soda can holds 12 ounces of fluid, that a pound is equivalent to 16 dry ounces, and that an ounce of water weighs approximately 1 dry ounce (at the temperature of the human body).

Conceptual Category: Algebra

In the Algebra conceptual category, students extend the work with expressions they started in the middle grades standards. They create and solve equations in context, utilizing the power of variable expressions to model real-world problems and solve them with attention to units and the meaning of the answers they obtain. They continue to graph equations, understanding the resulting picture as a representation of the points satisfying the equation. This conceptual category comprises a large portion of the Algebra I course and along with the Functions category represents the main body of content.

The Algebra category in higher mathematics is very closely allied with the Functions category (Progressions, Algebra 2012, 2):

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the expression is of the form $y = (\text{expression in } x)$ or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent they define the same function.
- The solutions to an equation in one variable can be understood as the input values that yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing functions defined by each side and finding the points where the graphs intersect.

Thus, in light of understanding functions, the main content of the Algebra category, solving equations, working with expressions, etc., has a very important purpose.

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P*. ★

An expression can be viewed as a recipe for a calculation, with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Students began this work in grades six and seven, and continue this work with more complex expressions. The following example might arise in a modeling context and emphasizes the importance of understanding the meaning of expressions in a given problem.

Example. A company uses two different-sized trucks to deliver sand. The first truck can transport *x* cubic yards, and the second *y* cubic yards. The first truck makes *S* trips to a job site, while the second makes *T* trips. What do the following expressions represent in practical terms? a. $S + T$ b. $x + y$ c. $xS + yT$ d. $\frac{xS+yT}{S+T}$	**Solutions.** a. $S + T =$ the total number of trips both trucks make to a job site. b. $x + y =$ the total amount of sand that both trucks can transport together. c. $xS + yT =$ the total amount of sand (in cubic yards) being delivered to a job site by both trucks. d. $\frac{xS+yT}{S+T} =$ the average amount of sand being transported per trip.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Linear, and exponential (integer inputs only); for A.CED.3, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA★**
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

An equation is a statement of equality between two expressions. The values that make the equation true are the solutions to the equation. An identity, in contrast, is true for all values of the variables; rewriting an expression in an equivalent form often creates identities. The solutions of an equation in one variable form a set of numbers which can be plotted on a number line; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. In this set of standards (A-CED.1–A.-CED.4), students create equations to solve problems, they correctly graph the equations on the coordinate plane, and they interpret solutions in a modeling context. The example below is designed to make students think about the meaning of the quantities presented in the context and choose which ones are appropriate for the two different constraints presented. In particular, note that the purpose of the task is to have students *generate* the constraint equations for each part (though the problem statements avoid using this particular terminology), and not to have students *solve* said equations (Illustrative Mathematics 2013).

Example: The coffee variety *Arabica* yields about 750 kg of coffee beans per hectare, while *Robusta* yields about 1200 kg per hectare. Suppose that a plantation has a hectares of *Arabica* and r hectares of *Robusta*.	**Solution and Comments:**
a. Write an equation relating a and r if the plantation yields 1,000,000 kg of coffee.	a. We see that a hectares of *Arabica* will yield $750a$ kg of beans, and that r hectares of *Robusta* will yield $1200r$ kg of beans. So the constraint equation is $750a + 1200r = 1{,}000{,}000.$
b. On August 14, 2003, the world market price of coffee was \$1.42 per kg of *Arabica* and \$0.73 per kg of *Robusta*. Write an equation relating a and r if the plantation produces coffee worth \$1,000,000.	b. We know that a hectares of *Arabica* yield $750a$ kg of beans worth \$1.42/kg for a total dollar value of $1.42(750a) = 1065a$. Likewise, r hectares of *Robusta* yield $1200r$ kg of beans worth \$0.73/kg for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of a and r coming from the total market value of the coffee is

	$1065a + 876r = 1{,}000{,}000.$

One change in the CA CCSSM is the creation of equations involving absolute values (A-CED.1). The basic absolute value function has at least two very useful definitions, a descriptive, verbal definition, and a formula definition. A common definition of the absolute value of x is:

$|x| =$ the distance from the number x to 0 (on a number line).

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid "formula" definition of absolute value reads:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

In other words, $|x|$ is simply x whenever x is 0 or positive, but $|x|$ is the opposite of x whenever x is negative. Either definition can be extended to an understanding of the expression $|x - a|$ as the distance between x and a on a number line, an interpretation that has many uses. For an application of this idea, suppose a certain bolt is to be mass-produced in a factory with the specification that its width should be 5mm with an error no larger than 0.01mm. If w represents the width of a given bolt produced on the line, then we want w to satisfy the inequality $|w - 5| \leq 0.01$, i.e., the difference between the actual width w and the target width should be less than or equal to 0.01 (MP.4, MP.6). Students should become comfortable with the basic properties of absolute values (e.g. $|x| + a \neq |x + a|$ in general, etc.), and with solving absolute value equations and interpreting the solution.

In higher math courses intervals on the number line are often denoted by an inequality of the form $|x - a| \leq d$ for a positive number d. For example $|x - 2| \leq \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This can be seen by interpreting $|x - 2| \leq \frac{1}{2}$ as "the distance from x to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.	On the other hand, in the case that $x - 2 < 0$, we would have $|x - 2| = -(x - 2) \leq \frac{1}{2}$, so that $x \geq 1\frac{1}{2}$. In the case that $x - 2 \geq 0$, we have $|x - 2| = x - 2 \leq \frac{1}{2}$ which means that $x \leq 2\frac{1}{2}$. Since we are looking for all x that satisfy both inequalities, the interval is $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear, learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = 1/16$.]

3.1 **Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA**

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. In Mathematics I, students solve linear equations and inequalities in one variable, including ones with absolute values and equations with coefficients represented by letters (A-REI.3, A-REI.3.1). When solving equations, students make use of the symmetric and transitive properties, and certain properties of equality with regards to operations (e.g. “equals added to equals is equal”). Standard A-REI.1 requires that in any situation, students can solve an equation *and explain the steps* as resulting from previous true equations and using the aforementioned properties (MP.3). In this way, the idea of *proof*, while not explicitly named, is given a prominent role in the solving of equations, and the reasoning and justification process is not simply relegated to a future mathematics course. The examples below illustrate the justification process that may be expected in Mathematics I.

On Solving Equations (Progressions, Algebra 2012, 10): A written sequence of steps is code for a narrative line of reasoning that would use words like “if”, “then”, “for all” and “there exists.” In the process of learning to solve equations, students should learn certain “if-then” moves: e.g. “if $x = y$ then $x + c = y + c$ for any c.” The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult	**Fragments of Reasoning** $2x - 5 = 16 - x$ $2x - 5 + x = 16 - x + x$ $3x - 5 = 16$ $3x = 21$ $x = \frac{21}{3} = 7$ This sequence of equations is short-hand for a line of reasoning: “*If twice a number minus 5 equals 16*

to detect incorrect or made-up moves later on. Thus the first requirement in this domain (A.-REI) is that students understand that solving equations is a process of reasoning (A.REI.1).	*minus that number, then three of that number minus 5 must be 16 by the properties of equality. But that means three times that number is 21, so the number is 7.*”

The solution techniques for solving equations can also be used to rearrange formulas to highlight specific quantities and explore relationships among the variables involved. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1+b_2}{2}\right)h$, can be solved for *h* using the same deductive process (MP.7, MP.8). As will be discussed later, functional relationships can often be explored more deeply by rearranging equations that define such relationships; thus, the ability to work with equations with letters as coefficients is an important skill.

Reasoning with Equations and Inequalities **A-REI**

Solve systems of equations. [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. The process of adding one equation to another is understood as, “if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum (or difference) of the left sides of the two equations is equal to the sum (or difference) of the right sides.” The reversibility of these steps justifies that we achieve an equivalent system of equations by doing this. This crucial point should be consistently noted when reasoning about solving systems of equations (Progressions, Algebra 2012, 11).

When solving systems of equations, students also make frequent use of the substitution property of equality, e.g., when solving the system $2x - 9y = 5$ and $y = \frac{1}{3}x + 1$ by substituting the expression $\frac{1}{3}x + 1$ for y in the first equation, to obtain

$2x - 9\left(\frac{1}{3}x + 1\right) = 5$. Students also solve such systems approximately, by using graphs and tables of values (A-REI.5-6). When presented in context, the method of solving a system of equations by elimination takes on meaning, as the example below shows.

Example: *Solving simple systems of equations.* To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret the system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve the system

$$3x + y = 40$$
$$4x + 2y = 58$$

Now consider the following scenario:

- Suppose 3 CD's and a magazine cost $40, while 4 CD's and 2 magazines cost $58.

And the questions

- What happens to the price when you add 1 CD and 1 magazine to your purchase?
- What is the price if you decided to buy only 2 CD's and no magazine?

Answering these questions amounts to realizing that since $(3x + y) + (x + y) = 40 + 18$, we must have that $x + y = 18$. Therefore, $(3x + y) + (-1)(x + y) = 40 + (-1)18$, which implies that $2x = 22$, so $x = 11$, or 1 CD costs $11.

The value of y can now be found using either of the original equations: $y = 7$, or a magazine costs $7.

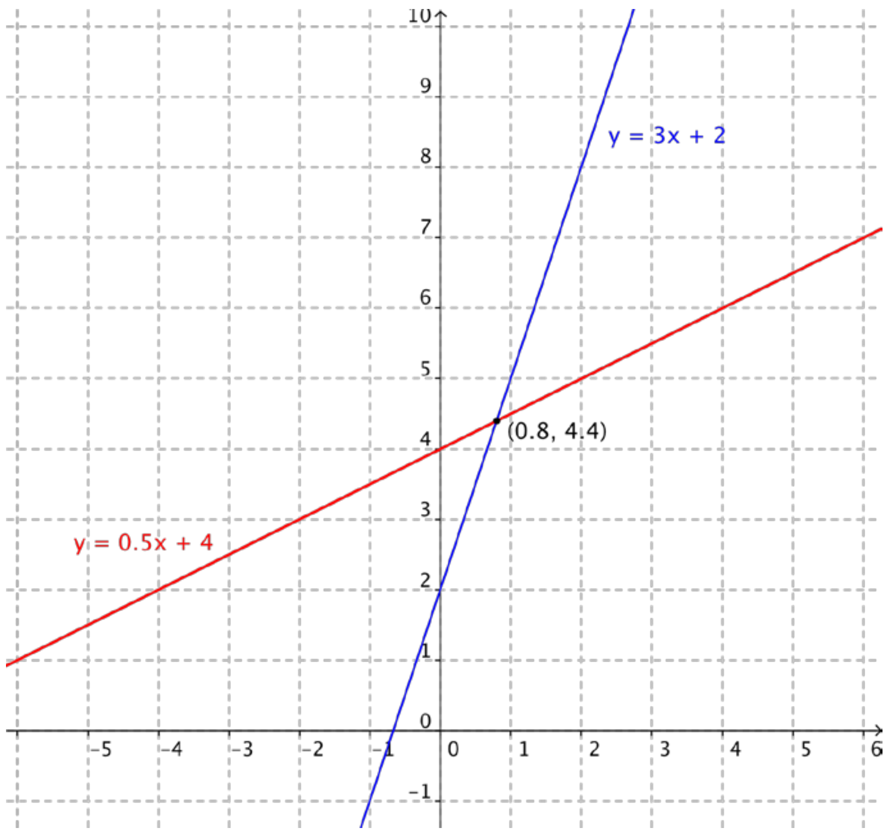
Reasoning with Equations and Inequalities A-REI

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

One of the most important goals of instruction in mathematics is to illuminate connections between various mathematical concepts. In particular, in standards A-REI.10-12, students learn the relationship between the algebraic representation of an equation and its graph plotted in the coordinate plane, and understand geometric interpretations of solutions to equations and inequalities. As students become more comfortable with function notation after studying standards F-IF.1–2, e.g. writing

$f(x) = 3x + 2$ and $g(x) = -\frac{1}{2}x + 4$, they begin to see solving the equation $3x + 2 = -\frac{1}{2}x + 4$ as solving the equation $f(x) = g(x)$; that is, they are finding those x-values where two functions take on the same output value. Moreover, they graph the two equations, and see that the x-coordinate(s) of the point(s) of intersection of the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the original equation (shown in the figure below).



Students also create tables of values for functions to approximate or find exact solutions to equations such as that above. For example, they may use spreadsheet software to construct a table such as the one below.

x	f(x)=3x+2	g(x)=-(.5)x+4
-3	-7	5.5
-2.5	-5.5	5.25
-2	-4	5
-1.5	-2.5	4.75
-1	-1	4.5
-0.5	0.5	4.25
0	2	4
0.5	**3.5**	**3.75**
1	**5**	**3.5**
1.5	6.5	3.25
2	8	3
2.5	9.5	2.75
3	11	2.5
3.5	12.5	2.25
4	14	2
4.5	15.5	1.75

Students can reason using this table that since $f(x) = 3.5$ at $x = .5$ and f is increasing, and $g(x) = 3.5$ at $x = 1$ and g is decreasing, the two functions must take on the same value somewhere between these values (MP.3, MP.6). In this example, since the original equation is of degree one, we know that there is only one solution, and using finer increments in x can approximate the solution. Examining graphs and tables along with solving equations algebraically helps students make connections between these various representations of functions and equations.

Conceptual Category: Geometry

In the standards for grades seven and eight, students began to see two-dimensional shapes as part of a generic plane (the Euclidean Plane) and began to explore transformations of this plane as a way to determine whether two shapes are congruent or similar. In Mathematics I, these notions are formalized and students use transformations to prove geometric theorems about triangles. Students then apply these triangle congruence theorems to prove other geometric results, engaging throughout in MP 3.

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

In the Geometry conceptual category, the commonly held—but imprecise—definition that shapes are congruent when they “have the same size and shape” is replaced by a more mathematically precise one: *two shapes are congruent if there is a sequence of rigid motions in the plane that takes one shape exactly onto the other*. This definition has been explored intuitively in grade eight, but is now investigated more closely. Earlier, students experimented with transformations in the plane, but now, students build more precise definitions for the *rigid motions*, rotation, reflection, and translation, based on previously defined and understood terms, such as angle, circle, perpendicular line, point, line, between, etc. (G-CO.1, 3, 4). Students base their understanding of these definitions on their experience with transforming figures using patty paper, transparencies, or geometry software, (G-CO.2, 3, 5), something they started doing in the grade eight standards. These transformations should be investigated both in a general plane as well as on a coordinate system—especially when explicitly describing transformations using precise names of points, translation vectors, and specific lines.

Example: *Defining Rotations*. Mrs. B wants to help her class understand the following definition of a *rotation:*

> A *rotation* about a point P through angle α is a transformation $A \longmapsto A'$ such that (1) if point A is different from P, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point A is the same as point P, then $A' = A$.

She gives her students a handout with several geometric shapes on it and a point P indicated on the page. In pairs, students are to copy the shapes onto a transparency sheet and rotate them through various angles about P. Students then transfer the rotated shapes back onto the original page, and measure various lenghts and angles as indicated in the definition.

While justfiying that the properties of the definition hold for the shapes she has given them, the students also make some observations about the effects of a rotation on the entire plane, for instance that:

- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

Later, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points P, both part of the object and not.

In standards G-CO.6-8, geometric transformations are given a more prominent role in the higher mathematics geometry curriculum than perhaps ever before. The new definition of congruence in terms of rigid motions applies to *any* shape in the plane, whereas previously congruence seemed to depend on criteria that were specific only to certain shapes. For example, the side-side-side (SSS) congruence criterion for triangles did not extend to quadrilaterals, seemingly suggesting that congruence was a notion specific to the shape one was considering. While it is true that there are specific criteria for determining congruence of certain shapes, the *basic notion of congruence* is the same for all shapes. In the common core, the SSS criteria for triangle congruence is a *consequence* of the definition of congruence, just as is the fact that if two polygons are congruent, then their sides and angles can be put into a correspondence such that each corresponding pair of sides and angles is congruent. This concept comprises the content of standards G-CO.7 and G.CO.8, which derive congruence criteria for triangles from the new definition of congruence.

A further discussion of standards G-CO.7 and G-CO.8 is warranted here. Standard G-CO.7 explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent. The depth of reasoning here is fairly substantial at this level; put in other words, students must be able to show using rigid motions that congruent triangles have congruent corresponding parts, and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one triangle to the other. The second statement may be more difficult to justify than the first for most students, so we present a justification of it here. Suppose we have two triangles ΔABC and ΔDEF such that the correspondence $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle was drawn on a fixed piece of paper, and the other drawn on a separate transparency, then a student could illustrate a translation, T, that takes point A to point D. A simple rotation R about the point A, if necessary, takes point B to point E, which we can be certain will occur since $AB \cong DE$ and rotations preserve lengths. Finally, the last step that may be needed is a reflection S about the side $\overline{AB}$, to take point C to point F. It is nontrivial why the image of point C is actually F. Since $\angle A$ is reflected about line $\overleftrightarrow{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overleftrightarrow{DF}$, since $\angle A \cong \angle D$. But since $AC \cong DF$, it must be the case that the image of point C coincides with F. The previous discussion amounts to the fact that the sequence of rigid motions, T, followed by R, followed by S, maps ΔABC exactly onto ΔDEF. Therefore, if we know that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. See the figure for an illustration of the steps in this reasoning.

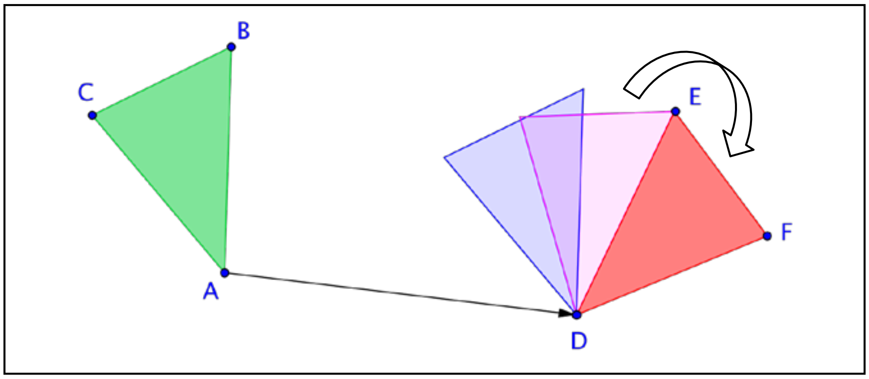


Figure 2: Illustration of the reasoning that corresponding parts of two triangles being congruent implies triangle congruence, in which point *A* is translated to *D*, the resulting image of ΔABC is rotated so as to place *B* onto *E*, and finally, the image is then reflected along line segment *AB* to match point *C* to *F*.

Similar reasoning applies for standard G-CO.8, in which students justify the typical triangle congruence criteria such as ASA, SAS and SSS. Experimentation with transformations of triangles where only two of the criteria are satisfied will result in counterexamples, and geometric constructions of triangles of prescribed side lengths, for example, in the case of SSS, will leave little doubt that any triangle constructed with these side lengths will be congruent to another, and that therefore SSS holds. Note that in standards G-CO.1-8, formal proof is not required. Students are asked to show using transformations that certain results are true.

Expressing Geometric Properties with Equations G-GPE

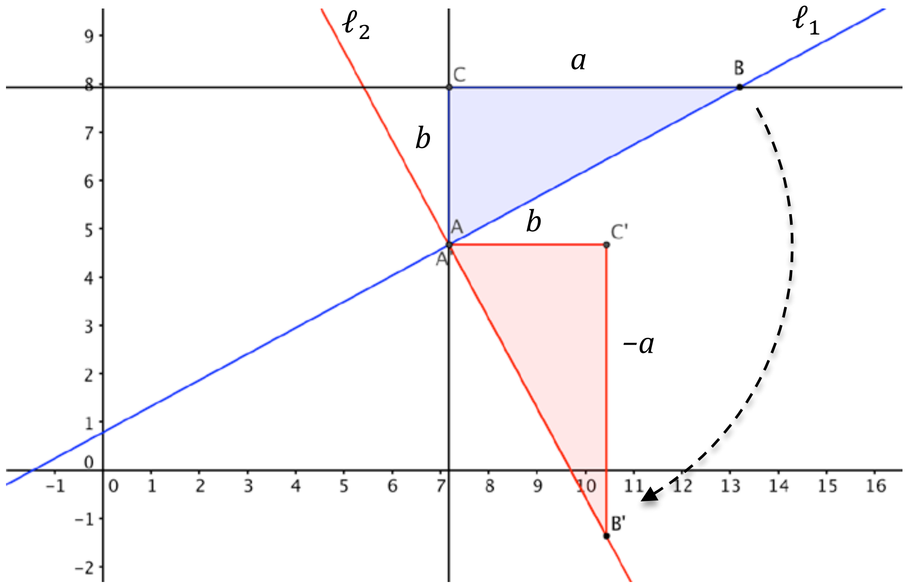
Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

The intersection of algebra and geometry is explored in this cluster of standards. In G.GPE.4, students use coordinates to prove simple geometric theorems, for instance, proving a figure defined by four points is a rectangle by proving that lines containing opposite sides of the figure are parallel and lines containing adjacent sides are perpendicular. Students must be fluent in finding slopes and equations of lines (where necessary) and understand the relationships between the slopes of parallel and perpendicular lines in order to do so (G-GPE.5).

While many simple geometric results can be proved algebraically, perhaps two results of high importance are the slope criteria for parallel and perpendicular lines themselves. Students have studied lines and linear equations since grade seven; here, they not only use relationships between slopes of parallel and perpendicular lines to solve problems, but they justify why they work. An intuitive argument for why parallel lines have the same slope might read: "Since the two lines never meet, each line must *keep up* with the other as we travel along the slopes of the lines. So it seems obvious that their slopes must be equal." This intuitive thought leads us to the equivalent statement: if given a pair of linear equations $\ell_1: y = m_1 x + b_1$ and $\ell_2: y = m_2 x + b_2$ (for $m_1, m_2 \neq 0$) such that $m_1 \neq m_2$, that is, such that *their slopes are different*, then the lines must intersect. Solving for the intersection of the two lines yields the x-coordinate of their intersection to be $x = \frac{b_2 - b_1}{m_1 - m_2}$ which surely exists since $m_1 \neq m_2$. The reasoning here is important: both understanding the steps of the argument and understanding why proving this statement is equivalent to proving the statement, "if $\ell_1 \parallel \ell_2$, then $m_1 = m_2$."

In addition, students are expected to justify why the slopes of two non-vertical, perpendicular lines ℓ_1 and ℓ_2 satisfy the relationship $m_1 = -\frac{1}{m_2}$, or $m_1 \cdot m_2 = -1$. While there are perhaps several ways to do this, one is shown here. Let ℓ_1 and ℓ_2 be any two non-vertical perpendicular lines. Let A be the intersection of the two lines, and let B be any other point on ℓ_1 above A. We draw a vertical line through A, a horizontal line through B, and we let C be the intersection of those two lines. ΔABC is a right triangle. If a is the horizontal displacement Δx from C to B and b is the length of $\overline{AC}$, then the slope of ℓ_1 is $m_1 = \frac{\Delta y}{\Delta x} = \frac{b}{a}$. By rotating ΔABC clockwise around A by 90 degrees, the hypotenuse $\overline{AB'}$ of the rotated triangle $\Delta AB'C'$ lies on ℓ_2. Using the legs of $\Delta AB'C'$, we see that the slope of ℓ_2 is $m_2 = \frac{\Delta y}{\Delta x} = \frac{-a}{b}$. Thus $m_1 \cdot m_2 = \frac{b}{a} \cdot \frac{-a}{b} = -1$. See the figure for an illustration of this proof (MP.1, MP.7).



The previous proofs make use of several ideas that students have learned about in this and prior courses, including the relationship between equations and their graphs in the plane (A-REI.10) and solving equations with variable coefficients (A-.REI.3). An investigative approach that at first explores several examples of lines that are perpendicular and their equations to find points, construct triangles, and decide if the triangles formed are right triangles will help ramp up to the second proof (MP.8). However, once more, the reasoning required to make sense of such a proof and to communicate the essence of it to a peer is an important goal of geometry instruction (MP.3).

Conceptual Category: Statistics and Probability

In Mathematics I, students build on their understanding of key ideas for describing distributions—shape, center, and spread—presented in the standards for grades six through eight. This enhanced understanding allows them to give more precise answers to deeper questions, often involving comparisons of data sets.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

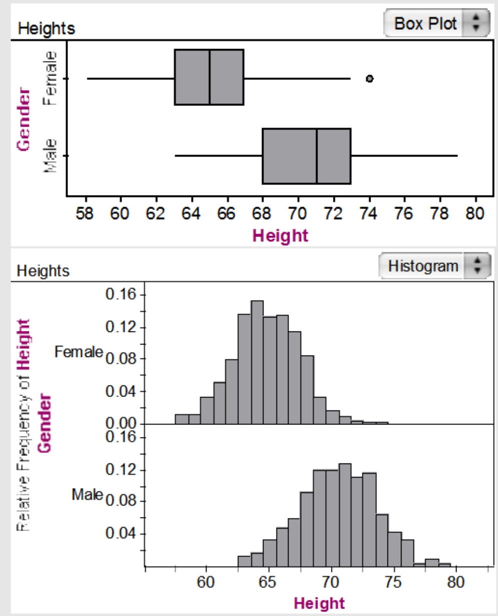
Standards S-ID.1-6 can be considered supporting standards with regards to Standards S-.ID.7-9, in the sense that they extend concepts students began learning in grades six through eight. Students use shape and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. Students may use parallel box-plots or histograms to compare differences in the shape, center and spread of comparable data sets (S-ID.1, 2).

Example (Progressions, High School Statistics and Probability 2012). The graphs below show two ways of comparing height data for males and females in the 20-29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called parallel (or side-by-side) box plots and parallel histograms (S-ID.1). The parallel box plots show an obvious difference in the medians and the questions about it just from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean (S.ID.2, S.ID.3). Students also observe that the two measures of center, median and mean, tend to be close to each other for symmetric distributions.

interquartile ranges (IQRs) for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 4 inches and 5 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer

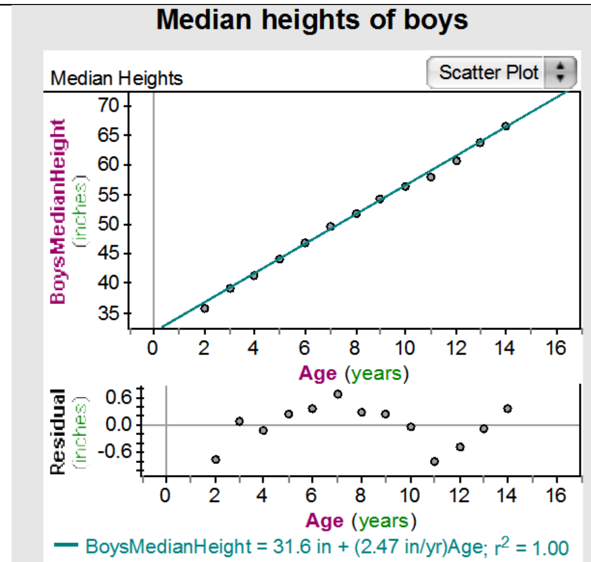
Comparing heights of males and females

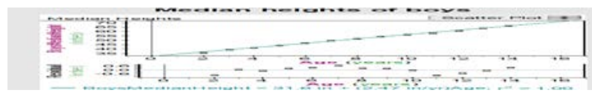


Heights of U.S. males and females in the 20–29 age group.
Source: *U.S. Census Bureau, Statistical Abstract of the United States: 2009*, Table 201.

Students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data. (S-ID.5, 6) Students have seen scatter plots in the grade eight standards and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model tells us about the relationship. Students must learn to take a careful look at scatter plots, as sometimes the “obvious” pattern does not tell the whole story and may be misleading. A line of best fit may appear to fit data almost perfectly, while an examination of the *residuals*, the collection of differences between corresponding coordinates on a least squares line and the actual data value for a variable may reveal more about the behavior of the data.

Example. Students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story, and can even be misleading. The graphs show the median heights of growing boys through the ages 2 to 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit (S-ID.6c). But, the *residuals*, the collection of differences between the corresponding coordinate on the least squares line and the actual data value for each age, reveal additional information. A plot of the residuals shows that growth at a constant rate does not proceed at a constant rate over those years (S.ID.6b).





Finally, students extend their work from topics covered in the grade eight standards and other topics in Mathematics I to interpret the parameters of a linear model in the context of data that it represents. They compute *correlation coefficients* using technology and interpret the value of the coefficient (MP.4, MP.5). Students see situations where correlation and causation are mistakenly interchanged, and they are careful to closely examine the story that data and computed statistics are trying to tell. (S-ID.7-9).

Mathematics I Overview

Number and Quantity

Quantities

- Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Geometry

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Make geometric constructions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Expressing Geometric Properties with Equations

- Use coordinates to prove simple geometric theorems algebraically.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making
(+) Indicates additional mathematics to prepare students for advanced courses

Mathematics I

Number and Quantity

Quantities N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P*. ★

Creating Equations A-CED

Create equations that describe numbers or relationships. [Linear, and exponential (integer inputs only); for A.CED.3, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA★**
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear, learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. **.** [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = 1/16$.]

3.1 **Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA**

Solve systems of equations. [Linear systems.]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions F-IF

Understand the concept of a function and use function notation. [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.).
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.).

Interpret functions that arise in applications in terms of the context. [Linear and exponential, (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear and exponential.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [For F.BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Interpret expressions for functions in terms of the situation they model. [Linear and exponential of form $f(x) = b^x + k$]

5. Interpret the parameters in a linear or exponential function in terms of a context.★

Geometry

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Expressing Geometric Properties with Equations G-GPE

Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★

3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. ★
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