Algebra II

Introduction

The purpose of this course is to extend students' understanding of functions and the real numbers, and to increase the tools students have for modeling the real world. They extend their notion of number to include complex numbers and see how the introduction of this set of numbers yields the solutions of polynomial equations and the Fundamental Theorem of Algebra. Students deepen their understanding of the concept of function, and apply equation-solving and function concepts to many different types of functions. The system of polynomial functions, analogous to the integers, is extended to the field of rational functions, which is analogous to the rational numbers. Students explore the relationship between exponential functions and their inverses, the logarithmic functions. Trigonometric functions are extended to all real numbers, and their graphs and properties are studied. Finally, students' statistics knowledge is extended to understanding the normal distribution, and they are challenged to make inferences based on sampling, experiments, and observational studies.

The standards in the traditional Algebra II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The content of the course will be expounded on below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off a list during isolated units of instruction, but rather content that should be present throughout the school year through rich instructional experiences.

What Students learn in Algebra II

Overview

Building on their work with linear, quadratic, and exponential functions, in Algebra II students extend their repertoire of functions to include polynomial, rational, and radical

The *Mathematics Framework* was adopted by the California State Board of Education on November 6, 2013. The *Mathematics Framework* has not been edited for publication.

functions.[1] Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. Based on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data— including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

Examples of Key Advances from Previous Grades or Courses

- In Algebra I, students added, subtracted and multiplied polynomials. In Algebra II, students divide polynomials with remainder, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle school algebra continue and deepen during high school. As early as grade 6, students began thinking about solving equations as a process of reasoning (6.EE.5). This perspective continues throughout Algebra I and Algebra II (A-REI).4 “Reasoned solving” plays a role in Algebra II because the equations students encounter can have extraneous solutions (A-REI.2).

[1] In this course rational functions are limited to those whose numerators are of degree at most 1 and denominators of degree at most 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials (CCSSI 2010).

- In Algebra I, students worked with quadratic equations with no real roots. In Algebra II, they extend the real numbers to complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicities) two roots in the complex numbers.
- In grade 8, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In Geometry, students proved theorems using coordinates (G-GPE.4–7). In Algebra II, students will build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (e.g., G-GPE.1).
- In Geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes. (PARCC 2012)

Connecting Standards for Mathematical Practice and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics*, and to the extent possible, content instruction should include attention to appropriate practice standards. There are ample opportunities for students to engage in each mathematical practice in Algebra II; the table below offers some general examples.

Standards for Mathematical Practice *Students…*	Examples of each practice in Algebra II
MP1. Make sense of	Students apply their understanding of various functions to real-world

problems and persevere in solving them.	problems. They approach complex mathematics problems and break them down into smaller-sized chunks and synthesize the results when presenting solutions.
MP2. Reason abstractly and quantitatively.	Students deepen their understanding of variable, for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real world context.
MP3. Construct viable arguments and critique the reasoning of others. ***Students build proofs by induction and proofs by contradiction. CA 3.1*** *(for higher mathematics only).*	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function to model a real world situation.
MP4. Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and examining patterns in data from real world contexts.
MP5. Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP6. Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of the complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP7. Look for and make use of structure.	Students see the operations of the complex numbers as extensions of the operations for real numbers. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP8. Look for and express regularity in repeating reasoning.	Students observe patterns in geometric sums, e.g. that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written: $1=2^1-1; 1+2=2^2-1;$ $1+2+4=2^3-1; 1+2+4+8=2^4-1$, and use this observation to make a conjecture about any such sum.

MP standard 4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Though the Modeling category has no specific standards listed within it, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Readers will see some standards marked with a star symbol (★) to indicate that they are *modeling standards*, that is, they present an opportunity for applications to real world modeling situations more so than other standards.

Examples of places where specific MP Standards can be implemented in the Algebra II standards will be noted in parentheses, with the standard(s) indicated.

Algebra II Content Standards by Conceptual Category

The Algebra II course is organized by conceptual category, domains, clusters, and then standards. Below, the overall purpose and progression of the standards included in Algebra II are described according to these conceptual categories. Note that the standards are not listed in an order in which they should be taught. Standards that are considered to be new to secondary grades teachers will be discussed in more depth than others.

Conceptual Category: Modeling

Throughout the higher mathematics CA CCSSM, certain standards are marked with a (★) symbol to indicate that they are considered modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics to real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real world situation and challenged to ask a question, all sorts of new issues arise: which of the quantities present in this situation are known and unknown? Can I make a table of data? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g. exponential functions) but may find that a new formula or function will apply. They may see that solving an equation arises as a necessity when trying to answer their question and that oftentimes the equation arises as the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. This will be a new approach for many teachers and will be challenging to implement, but the effort will produce students who can appreciate that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

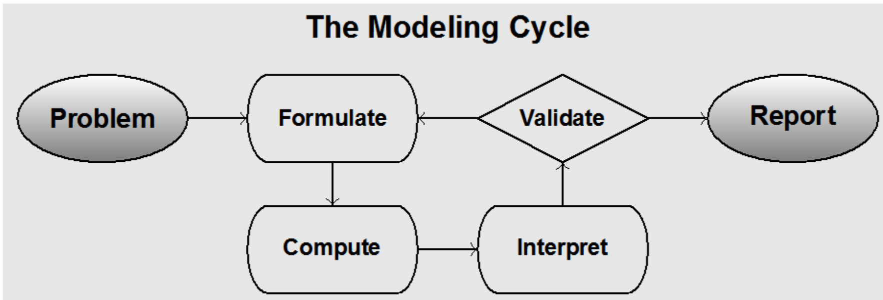


Figure 1: The modeling cycle. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret their results, validate them, and report out.

Throughout the Algebra II chapter, the included examples will be framed as much as possible as modeling situations, to serve as illustrations of the concept of mathematical modeling. The big ideas of polynomial and rational functions, graphing, trigonometric functions and their inverses, and applications of statistics will be explored through this lens. The reader is encouraged to consult the Appendix, "Mathematical Modeling," for a further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

Work on functions began in Algebra I. In Algebra II, students encounter more sophisticated functions, such as polynomial functions of degree greater than 2, exponential functions with the domain all real numbers, logarithmic functions, and extended trigonometric functions and their inverses. Several standards of the functions category are repeated here, illustrating that the standards attempt to reach depth of understanding of the *concept* of function. Students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary (The University of Arizona Progressions Documents for the Common Core Math

Standards [Progressions], Functions 2012, 7). For instance, in Algebra II students see quadratic, polynomial, and rational functions as belonging to the same system.

Interpreting Functions **F-IF**

Interpret functions that arise in applications in terms of the context. [Emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

In this domain students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots and other characteristics of functions is becoming more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (F-IF.4-9). The example problem below illustrates some of these standards.

Example. *The Juice Can.* Suppose we wanted to know the minimal surface area of a cylindrical can of a fixed volume. Here, we consider the surface	The data suggests that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 cm. Successive

area in units cm^2, the radius in units cm, and the volume to be fixed at 355 ml = 355 cm^3. One can find the surface area of this can as a function of the radius:

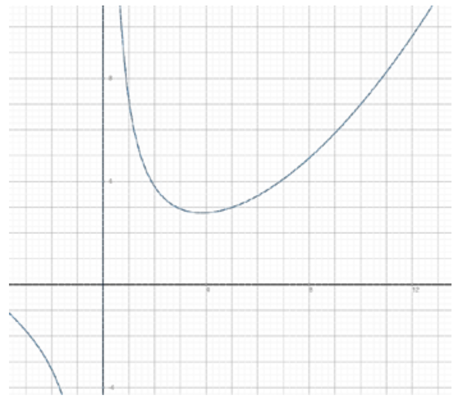
$$S(r) = \frac{2(355)}{r} + 2\pi r^2.$$

(See *The Juice Can Equation* example in the Algebra conceptual category.) This representation allows us to examine several things.

First, a table of values will give a hint at what the minimal surface area is. The table shown lists several values for S based on r:

r (cm)	S (cm^2)
0.5	1421.6
1.0	716.3
1.5	487.5
2.0	380.1
2.5	323.3
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	299.0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

approximation using values of r between these values will yield a better estimate. But how can we be sure that the minimum is truly located here? A graph of $S(r)$ can give us a hint:



Furthermore students can deduce that as r gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as r grows larger, so that there is truly a minimum somewhere in the interval $[3.5, 4.5]$. (F.IF.4, F.IF.5, F.IF.7-9)

Graphs help us reason about rates of change of functions (F.IF.6). Students learned in Grade 8 that the *rate of change* of a linear function is equal to the slope of its graph. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For nonlinear functions, however, rates of change are not constant, and so we talk about average rates of change over an interval. For example, for the function g defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points $(2, 4)$ and $(5, 25)$ on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (Progressions 2012, 9). Students could investigate similar rates of change over intervals for the Juice Can problem shown previously.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities.★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*★

Build new functions from existing functions. [Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.*

Students in Algebra II develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (F-BF.1). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations involving cooling or heating involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is $70°$ and a cup of tea is made with boiling water at a temperature of $212°$, a student can express the function describing the temperature as a function of time by using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, leading to a function of the form:

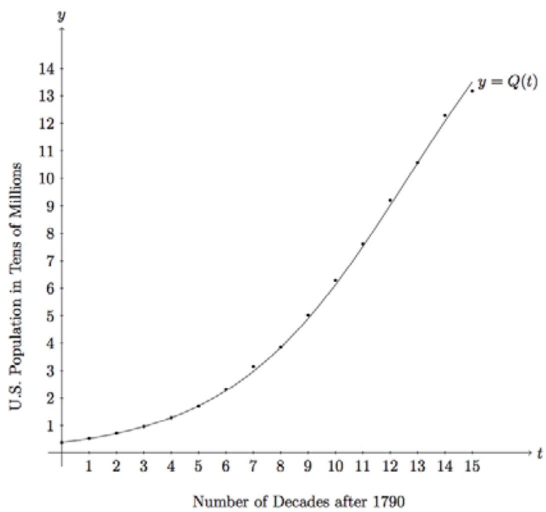
$$T(t) = 70 + 142e^{-kt}.$$

Students might determine the constant k experimentally. (MP.4, MP.5)

Example (Adapted from Illustrative Mathematics 2013). *Population Growth.* The approximate United States Population measured each decade starting in 1790 up through 1940 can be modeled by the function

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)},$$

where t represents decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.



Some possible questions:

a. According to this model for the U.S. population, what was the population in the year 1790?

b. According to this model, when did the population first reach 100,000,000? Explain.

c. According to this model, what should be the population of the U.S. in the year 2010? Find a prediction of the U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions: a. The population in 1790 is given by $P(0)$, which we easily find is $3{,}900{,}000$ since $e^{0.31(0)} = 1$.

b. This is asking us to find t such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 1,000,000 and dividing both sides of the equation by $100{,}000{,}000$ simplifies this equation to

$$\frac{3.9 \times 2 \times e^{.31t}}{200 + 3.9(e^{.31t} - 1)} = 1.$$

Using some algebraic manipulation and solving for t gives $t \approx \frac{1}{0.31}\ln 50.28 \approx 12.64$. This means it would take about 126.4 years after 1790 for the population to reach 100 million.

c. The population 22 decades after 1790 would be approximately 190,000,000, too low by about 119,000,000 from the estimated U.S. population of 309,000,000 in 2010.

d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{.31t}$. Thus, for very large t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{.31t}}{3{,}900{,}000e^{.31t}}$$
$$= 200{,}000{,}000$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

Students can make good use of graphing software to investigate the effects of replacing a function $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for different types of functions (MP.5). For example, starting with the simple quadratic function $f(x)=x^2$, students see the relationship between these transformed functions and the vertex-form of a general quadratic, $f(x)=a(x-h)^2+k$. They understand the notion of a *family of functions*, and characterize such function families based on their properties. These ideas will be explored further with trigonometric functions (F-TF.5).

In F-BF.4a, students learn that some functions have the property that an input can be recovered from a given output, i.e., the equation $f(x)=c$ can be solved for x, given that c lies in the range of f. They understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard F-LE.4 described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so this standard should be treated carefully as preparation for a deeper understanding.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology. ★[Logarithms as solutions for exponentials]

4.1 Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Students have worked with exponential models in Algebra I and further in Algebra II. Since the exponential function $f(x)=b^x$ is always increasing or always decreasing for $b \neq 0,1$, we can deduce that this function has an inverse, called *the logarithm to the base* b, denoted by $g(x)=\log_b x$. The logarithm has the property that $\log_b x = y$ if and only if $b^y = x$, and arises in contexts where one wishes to solve an exponential equation. Students find logarithms with base b equal to 2, 10, or e, by hand and using technology (MP.5). In F.LE.4.1-4.3, students explore the properties of logarithms, such as that $\log_b xy = \log_b x + \log_b y$, and connect these properties to those of exponents (e.g., the previous property comes from the fact that the logarithm is representing an exponent, and that $b^{n+m} = b^n \cdot b^m$). Students solve problems involving exponential functions and logarithms and express their answers using logarithm notation (F-LE.4). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; opportunities for instruction should emphasize this relationship.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Prove and apply trigonometric identities.

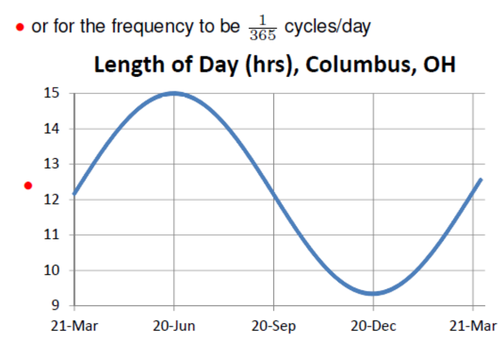
8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

In this set of standards, students expand on their understanding of the trigonometric functions first developed in Geometry. At first, the trigonometric functions apply only to angles in right triangles; $\sin\theta$, $\cos\theta$, and $\tan\theta$ only make sense for $0 < \theta < \frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, we see that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of θ that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, we take $\cos\theta$ to be the x-coordinate of the point corresponding to this rotation and $\sin\theta$ to be the y-coordinate of this point. This interpretation of sine and cosine immediately yield the Pythagorean Identity: that $\cos^2\theta + \sin^2\theta = 1$. This basic identity yields others through algebraic manipulation, and allows one to find values of other trigonometric functions for a given θ if one of them is known (F-TF.1, 2, 8).

The graphs of the trigonometric functions should be explored with attention to the connection between the unit circle representation of the trigonometric functions and their properties, e.g., to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeroes, etc. In standard F-TF.5, students use trigonometric functions to model periodic phenomena. Connected to standard F-BF.3 (families of functions), they begin to understand the relationship between the parameters appearing in the general cosine function $f(x) = A \cdot \cos(Bx - C) + D$ (and sine function) and the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry).

Example (Progressions, Functions 2012, 19): *Modeling Daylight Hours.* By looking at data for length of days in Columbus, OH, students see that day length is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the different of the maximum and minimum. We set $A = 12.17$ and $B = 2.83$ as approximations of these values. With some support, students determine that for the period to be 365 days (per cycle), $C = 2\pi/365$ and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83 \sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t the day of the year from March 21. Considering questions such as when to plant a garden, i.e. when there are at least 7 hours of midday sunlight, students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$, and find that May 1 and August 10 bookend this interval of time.



Students can investigate many other trigonometric modeling situations such as simple predator-prey models, sound waves, and noise cancellation models.

Conceptual Category: Number and Quantity

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

In Algebra I, students worked with examples of quadratic functions and solving quadratic equations, where they encountered situations in which a resulting equation did not have a solution that is a real number, e.g. $(x-2)^2 = -25$. In Algebra II, students complete their extension of the concept of number to include complex numbers, numbers of the form $a + bi$, where i is a number with the property that $i^2 = -1$. Students begin to work with complex numbers and apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations like those above, by finding square roots of negative numbers: e.g. $\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25} \cdot \sqrt{-1} = 5i$ (MP.7). They also apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations like those above:

$$(x-2)^2 = -25, \text{ which implies } |x-2| = 5i, \text{ or } x = 2 \pm 5i.$$

Now equations like these have solutions, and the extended number system forms yet another system that behaves according to familiar rules and properties (N-CN.1-2, N-CN.7-9). By exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show the theorem is true for quadratic polynomials by an application of the quadratic formula and an understanding of the relationship between roots of a quadratic equation and the linear factors of the quadratic polynomial (MP.2).

Conceptual Category: Algebra

Along with the Number and Quantity standards in Algebra II, the Algebra conceptual category standards develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Rational numbers extend the arithmetic of integers by allowing division by all numbers except zero; similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this section is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

In Algebra II, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by "chunking" (i.e. viewing parts of an expression as a single entity) (A-SSE.1, 2). For example, their facility with using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

In a Physics course, students may encounter an expression such as $L_0\sqrt{1 - \frac{v^2}{c^2}}$, which arises in the theory of special relativity. They can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v = 0$ and equal to 0 when $v = c$—and furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (Progressions, Algebra 2012, 4).

By examining the sums of examples of finite geometric series, students can look for patterns to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{(1-r)}$. They may derive the formula, either with Proof by Mathematical Induction (MP3), or by other means (A-SSE.4), as shown in the example below.

Example. *Sum of a Geometric Series.* Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (MP5, MP.8). Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments like retirement accounts, finding total lottery payout prizes, and more (MP.4).

In general, a finite geometric series has the form:

$$\sum_{k=0}^{n} ar^k = a(1 + r + r^2 + \cdots + r^{n-1} + r^n).$$

If we denote by S the sum of this series, then some algebraic manipulation shows that

$$S - rS = a - ar^{n+1}$$

Applying the distributive property to the common factors and solving for S shows that

$$S(1-r) = a(1-r^{n+1}),$$

so that

$$S = \frac{a(1-r^{n+1})}{1-r}.$$

Students hone their ability to flexibly see expressions such as $A_n = A_0\left(1+\frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret

$$A_1 + A_2 + \cdots + A_{12} = 100\left(1+\frac{.15}{12}\right)^1 + 100\left(1+\frac{.15}{12}\right)^2 + \cdots 100\left(1+\frac{.15}{12}\right)^{12}$$

as a type of geometric series that would calculate the total value in an investment account at the end of one year if we deposited \$100 at the beginning of each month (MP.2, MP.4, MP.7). They apply the formula for geometric series to find this sum.

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[2]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

In Algebra II, students continue developing their understanding of the set of polynomials as a system analogous to the set of integers that exhibits certain properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (A-APR.1-3). It is shown that when we divide a polynomial $p(x)$ by $(x - a)$, we are writing $p(x)$ in the following way:

$$p(x) = q(x) \cdot (x - a) + r,$$

where r is a constant. This can be done by inspection or by polynomial long division (A-APR.6). It follows that $p(a) = q(a) \cdot (a - a) + r = q(a) \cdot 0 + r = r$, so that $(x - a)$ is a factor of $p(x)$ if and only if $p(a) = 0$. This result is generally known as the Remainder Theorem (A.APR.2), and provides an easy check to see if a polynomial has a given linear polynomial as a factor. This topic should not be simply reduced to "synthetic division," which reduces the theorem to a method of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (MP.3) (Progressions, Algebra 2012, 7).

Students use the zeroes of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (A-APR.3). The notion that the polynomials can be used to approximate other functions is important in

2 The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

higher mathematics courses such as Calculus, and students can get a start here. Standard A.APR.3 is the first step in a progression that can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

In Algebra II, students explore rational functions as a system analogous to the rational numbers. They see rational functions as useful for describing many real-world situations, for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance d_0, and obtaining $r = \frac{d_0}{t}$. Now students see that any two polynomials can be divided in much the same way as with numbers (provided the divisor is not zero). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials. They should have opportunities to evaluate various rational expressions for many values of x, both by hand and using software, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. Developing an understanding of the behavior of rational expressions in this way helps students see them as functions, and sets the stage for working with simple rational functions.

Example. *The Juice Can.* If someone wanted to investigate the shape of a juice can of minimal surface area, they could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the formula:

$$S = 2\pi rh + 2\pi r^2,$$

since the two circular bases of the can contribute $2\pi r^2$units of surface area, while the outside surface of the can contributes an area in the shape of a rectangle with length the circumference of the base, $2\pi r$, and height equal to h. Since the volume is fixed, we can find h in terms of r: $h = \frac{V_0}{\pi r^2}$, and substitute this into the equation for the surface area:

$$S = 2\pi r \cdot \frac{V_0}{\pi r^2} + 2\pi r^2$$
$$= \frac{2V_0}{r} + 2\pi r^2.$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed. (See also A.CED.4, F.BF.4-9.)

In addition, students are able to rewrite rational expressions in the form $a(x) = q(x) \cdot b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary, e.g. to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in manipulating symbols properly, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above; e.g., rates of growth, approximation, roots, axis-intersections, asymptotes, end behavior, etc.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★

In Algebra II, students work with all available types of functions to create equations (A-CED.1). While functions used in A-CED.2, 3, and 4 will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Algebra I. For example, knowing how to find the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. For an example of standard A.CED.4, see the *Juice Can* problem earlier in this section.

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the *x*-coordinates of the points where the graphs of the equations *y* = *f*(*x*) and *y* = *g*(*x*) intersect are the solutions of the equation *f*(*x*) = *g*(*x*); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where *f*(*x*) and/or *g*(*x*) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Students extend their equation solving skills to those that involve rational expressions and radical equations; they make sense of extraneous solutions when they arise (A-REI.2). In particular, students understand that when solving equations, the flow is reasoning is generally forward, in the sense that we assume a number x is a solution of the equation and then find a list of possibilities for x. But not all steps in this process are reversible, e.g. while it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (Progressions, Algebra 2012, 10). Thus students understand that some steps are reversible and some are not, and anticipate extraneous solutions. In addition, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (A-REI.11), and understand that some equations can only be solved approximately with the tools they possess.

Conceptual Category: Geometry

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 **Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola, and graph the equation.** [In Algebra II, this standard addresses only circles and parabolas.] **CA**

No traditional Algebra II course would be complete without an examination of planar curves represented by the general equation $ax^2 + by^2 + cx + dy + e = 0$. In Algebra II, students use completing the square (a skill learned in Algebra I) to decide if the equation represents a circle or parabola. They graph the shapes and relate the graph to the equation. The study of ellipses and hyperbolas is reserved for a later course.

Conceptual Category: Statistics and Probability

Students in Algebra II move beyond analyzing data to making sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (S-IC.1) (Progressions, High School Statistics and Probability 2012). By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (S-IC.2). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. In Algebra II, students build on their understanding of data distributions to help see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (S-ID.4). In addition, they can learn through examples the *empirical rule*, that for a normally distributed data set, 68% of the data lies within one standard deviation of the mean, and that 95% are within two standard deviations of the mean.

Example. *The Empirical Rule.* Suppose that SAT mathematics scores for a particular year are approximately normally distributed with a mean of 510 and a standard deviation of 100.	**Solutions:**
a. What is the probability that a randomly selected score is greater than 610? b. Greater than 710? c. Between 410 and 710? d. If a students' score is 750, what is the students' percentile score (the proportion of scores below 750)?	a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32 or 0.16. The calculator gives 0.1586. b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05 or 0.025. The calculator gives 0.0227. c. The area under a normal curve from one standard deviation below the mean to two standard deviations above is about 0.815. The calculator gives 0.8186. d. Either using the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4) the calculator gives 0.9918.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment (CCSSI 2010). In standards S-IC.4 and 5, the focus should be on the variability of results from experiments—that is, focused on statistics as a way of dealing with, not eliminating, inherent randomness. Given that standards S-IC.1-6 are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example (Adapted from Progressions, High School Statistics and Probability 2012). *Estimating a Population Proportion.* Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 agree, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the *true* proportion of homeowners who favor the tax? How can we tell?	If we simulate this sampling situation (MP.4) using a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then she can get an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of .125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

As in Geometry, students apply probability models to make and analyze decisions. In Algebra II, this skill is extended to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false positive and false negative results. See the “High School Progression on Statistics and Probability” for more explanation and examples: http://ime.math.arizona.edu/progressions/.

Algebra II Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Geometry

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making
(+) Indicates additional mathematics to prepare students for advanced courses.

Algebra II

Number and Quantity

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[3]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

3. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Emphasize selection of appropriate models.]

[3] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities.★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*★

Build new functions from existing functions. [Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   - a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★[Logarithms as solutions for exponentials]

4.1 Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Geometry

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola, and graph the equation. [In Algebra II, this standard addresses only circles and parabolas.] **CA**

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★