1   # Grade Six

2

3 In the years prior to grade six, students built a foundation in numbers and operations,

4 geometry, and measurement and data. As they enter sixth grade, students are fluent in

5 addition, subtraction, and multiplication with multi-digit whole numbers and have a solid

6 conceptual understanding of all four operations with positive rational numbers, including

7 fractions. Students' understanding of measurement concepts (e.g., length, area,

8 volume, and angles) has solidly begun, and how to represent and interpret data is

9 emerging (Adapted from The Charles A. Dana Center Mathematics Common Core

10 Toolbox 2012).

11

12   ## WHAT STUDENTS LEARN IN GRADE SIX

13   [Note: Sidebar]

Grade Six Critical Areas of Instruction

In grade six, instructional time should focus on four critical areas: (1) connecting ratio, rate, and percentage to whole number multiplication and division and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; and (4) developing understanding of statistical thinking (CCSSO 2010, Grade 6 Introduction).

Students also work toward fluency with multi-digit division and multi-digit decimal operations.

14

15 **Grade Six Standards for Mathematical Content**

16 The Standards for Mathematical Content emphasize key content, skills, and practices at

17 each grade level and support three major principles:

18 • **Focus:** Instruction is focused on grade level standards.

19 • **Coherence:** Instruction should be attentive to learning across grades and should

20   link major topics within grades.

21 • **Rigor:** Instruction should develop conceptual understanding, procedural skill and

22   fluency, and application.

The *Mathematics Framework* was adopted by the California State Board of Education on November 6, 2013. *The Mathematics Framework* has not been edited for publication.

23   Grade level examples of focus, coherence and rigor will be indicated throughout the

24   chapter.

25

26   Not all of the content in a given grade is emphasized equally in the standards. Cluster

27   headings can be viewed as the most effective way to communicate the **focus** and

28   **coherence** of the standards. Some clusters of standards require a greater instructional

29   emphasis than the others based on the depth of the ideas, the time that they take to

30   master, and/or their importance to future mathematics or the later demands of college

31   and career readiness.

32

33   The following Grade 6 Cluster-Level Emphases chart highlights the content emphases

34   in the standards at the cluster level for this grade. The bulk of instructional time should

35   be given to "Major" clusters and the standards within them. However, standards in the

36   "Supporting" and "Additional" clusters should not be neglected. To do so will result in

37   gaps in students' learning, including skills and understandings they may need in later

38   grades. Instruction should reinforce topics in major clusters by utilizing topics in the

39   supporting and additional clusters. Instruction should include problems and activities

40   that support natural connections between clusters.

41

42   Teachers and administrators alike should note that the standards are not topics to be

43   checked off a list during isolated units of instruction, but rather content to be developed

44   throughout the school year through rich instructional experiences and presented in a

45   coherent manner (Adapted from the Partnership for Assessment of Readiness for

46   College and Careers [PARCC] 2012).

47

48   [**Note**: The Emphases chart should be a graphic inserted in the grade level section. The

49   explanation "key" needs to accompany it.]

50

51   **Grade 6 Cluster-Level Emphases**

52   **Ratios and Proportional Relationships**

53   •   **[m]:** Understand ratio concepts and use ratio reasoning to solve problems. (**6.RP.1-3▲**)

54

55 **The Number System**

56 • **[m]:** Apply and extend previous understandings of multiplication and division to divide
57 fractions by fractions. (**6.NS.1▲**)

58 • **[a/s]:** Compute fluently with multi-digit numbers and find common factors and multiples.
59 (**6.NS.2-4**)

60 • **[m]:** Apply and extend previous understandings of numbers to the system of rational
61 numbers. (**6.NS.5-8▲**)

62

63 **Expressions and Equations**

64 • **[m]:** Apply and extend previous understandings of arithmetic to algebraic expressions.
65 (**6.EE.1-4▲**)

66 • **[m]:** Reason about and solve one-variable equations and inequalities. (**6.EE.5-8▲**)

67 • **[m]:** Represent and analyze quantitative relationships between dependent and independent
68 variables. (**6.EE.9▲**)

69

70 **Geometry**

71 • **[a/s]:** Solve real-world and mathematical problems involving area, surface area, and
72 volume.[1] (**6.G.1-4**)

73

74 **Statistics and Probability**

75 • **[a/s]:** Develop understanding of statistical variability. **(6.SP.1-3)**

76 • **[a/s]:** Summarize and describe distributions. **(6.SP.4-5)**

77

Explanations of Major, Additional and Supporting Cluster-Level Emphases
Major[2] [m] (▲) clusters – areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness.
Additional [a] clusters – expose students to other subjects; may not connect tightly or explicitly to the major work of the grade **Supporting [s]** clusters – rethinking and linking; areas where some material is being covered, but in a way that applies core understanding; designed to support and strengthen areas of major emphasis.   *A Note of Caution: Neglecting material will leave gaps in students' skills and understanding and will leave students unprepared for the challenges of a later grade.

[1] In this cluster, students work on problems with areas of triangles and volumes of right rectangular prisms, which connect to work in the Expressions and Equations domain. In addition, another standard within this cluster asks students to draw polygons in the coordinate plane, which supports work with the coordinate plane in the Number System domain.
[2] The ▲ symbol will indicate standards in a Major Cluster in the narrative.
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78 (Adapted from Smarter Balanced Assessment Consortia [Smarter Balanced], DRAFT

79 Content Specifications 2012).

80

81 **Connecting Mathematical Practices and Content**

82 The Standards for Mathematical Practice (MP) are developed throughout each grade

83 and, together with the content standards, prescribe that students experience

84 mathematics as a rigorous, coherent, useful, and logical subject that makes use of their

85 ability to make sense of mathematics. The MP standards represent a picture of what it

86 looks like for students to understand and do mathematics in the classroom and should

87 be integrated into every mathematics lesson for all students.

88

89 Although the description of the MP standards remains the same at all grades, the way

90 these standards look as students engage with and master new and more advanced

91 mathematical ideas does change. Below are some examples of how the MP standards

92 may be integrated into tasks appropriate for Grade 6 students. (Refer to pages ## in the

93 Overview of the Standards Chapters for a complete description of the MP standards.)

94

95   **Standards for Mathematic Practice (MP)**

96   **Explanations and Examples for Grade Six**

Standards for Mathematical Practice	Explanation and Examples
MP.1. Make sense of problems and persevere in solving them.	In grade six, students solve real world problems through the application of algebraic and geometric concepts. These problems involve ratio, rate, area, and statistics. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves, "What is the most efficient way to solve the problem?", "Does this make sense?", and "Can I solve the problem in a different way?" Students can explain the relationships between equations, verbal descriptions, and tables and graphs. Mathematically proficient students check their answers to problems using a different method.
MP.2. Reason abstractly and	Students represent a wide variety of real world contexts through the use of rational numbers and variables in mathematical expressions, equations, and inequalities. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to operate with symbolic representations

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quantitatively.	by applying properties of operations or other meaningful moves. Teachers might ask, "How do you know" or "What is the relationship of the quantities?" to reinforce students' reasoning and understanding.
MP.3. Construct viable arguments and critique the reasoning of others.	Students construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, and graphs, tables, and other data displays (e.g., box plots, dot plots, histograms). They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions like "How did you get that?", "Why is that true?" and "Does that always work?" They explain their thinking to others and respond to others' thinking.
MP.4. Model with mathematics	In grade six, students model problem situations symbolically, graphically, in tables, contextually, and with drawings of quantities as needed. Students form expressions, equations, or inequalities from real world contexts and connect symbolic and graphical representations. Students begin to explore covariance and represent two quantities simultaneously. Students use number lines to compare numbers and represent inequalities. They use measures of center and variability and data displays (e.g., box plots and histograms) to draw inferences about and make comparisons between data sets. Students need many opportunities to connect and explain the connections between the different representations. They should be able to use any of these representations as appropriate to a problem context. Students should be encouraged to answer questions, such as "What are some ways to represent the quantities?" or "What formula might apply in this situation?"
MP.5. Use appropriate tools strategically.	Students consider available tools (including estimation and technology) when solving a mathematical problem and decide when certain tools might be helpful. For instance, students in grade six may decide to represent figures on the coordinate plane to calculate area. Number lines are used to create dot plots, histograms, and box plots to visually compare the center and variability of the data. Visual fraction models can be used to represent division of fractions situations. Additionally, students might use physical objects or applets to construct nets and calculate the surface area of three-dimensional figures. Students should be encouraged to answer questions such as, "What approach are you considering trying first?" or "Why was it helpful to use…?"
MP.6. Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to rates, ratios, geometric figures, data displays, and components of expressions, equations, or inequalities. When using ratio reasoning in solving problems, students are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. Students also learn to determine an appropriate degree of precision when working with rational

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	numbers in a situational problem. Teachers might ask "What mathematical language, definitions, properties…can you use to explain…?
MP.7. Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. For instance, students recognize patterns that exist in ratio tables recognizing both the additive and multiplicative properties. Students apply properties to generate equivalent expressions (e.g., $6 + 3x = 3(2 + x)$ by distributive property) and solve equations (e.g., $2c + 3 = 15, 2c = 12$ by subtraction property of equality, $c = 6$ by division property of equality). Students compose and decompose two- and three-dimensional figures to solve real world problems involving area and volume. Teachers might ask, "What do you notice when…?" or "What parts of the problem might you eliminate, simplify…?"
MP.8. Look for and express regularity in repeated reasoning.	In grade six, students use repeated reasoning to understand algorithms and make generalizations about patterns. During opportunities to solve and model problems designed to support generalizing, they notice that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ and construct other examples and models that confirm their generalization. Students connect place value and their prior work with operations to understand algorithms to fluently divide multi-digit numbers and perform all operations with multi-digit decimals. Students informally begin to make connections between covariance, rates, and representations showing the relationships between quantities. Students should be encouraged to answer questions, such as "How would we prove that…?" or "How is this situation like and different from other situations?"

97 (Adapted from Arizona Department of Education [Arizona] 2012, North Carolina

98 Department of Public Instruction [N. Carolina] 2012, and Georgia Department of

99 Education [Georgia] 2011)

100

101 **Standards-based Learning at Grade Six**

102 The following narrative is organized by the domains in the Standards for Mathematical

103 Content and highlights some necessary foundational skills from previous grades and

104 provides exemplars to explain the content standards, highlight connections to the

105 various Standards for Mathematical Practice (**MP**), and demonstrate the importance of

106 developing conceptual understanding, procedural skill and fluency, and application. A

107   triangle symbol (▲) indicates standards in the major clusters (refer to the Grade 6

108   Cluster-Level Emphases table on page #).

109

110   **Domain: Ratios and Proportional Relationships**

111   A critical area of instruction in grade six is to connect ratio, rate, and percentage to

112   whole number multiplication and division and use concepts of ratio and rate to solve

113   problems. Students' prior understanding of and skill with multiplication, division, and

114   fractions contribute to their study of ratios, proportional relationships, unit rates, and

115   percent in grade six. In grade seven these concepts will extend to include scale

116   drawings, slope, and real-world percent problems.

117

Ratios and Proportional Relationships	6.RP

Understand ratio concepts and use ratio reasoning to solve problems.
1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*
2. Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."[1]*

118

119   A *ratio* is a pair of non-negative numbers, $A:B$, which are not both zero.[3] In grade six,

120   students learn that ratios are a comparison of two numbers or quantities and that there

121   are two types of ratios—part-to-whole and part-to-part (**6.RP. 1▲**).

Types of Ratios in Sixth Grade
Part-to-whole ratios compare a specific part to the whole (e.g., the number of girls in the class,12, to the number of students in the class, 28, is the ratio 12 to 28, expressed as 12:28).
Part-to-part ratios compare two parts (e.g., the number of girls in the class, 12, compared to the number of boys in the class, 16, is the ratio 12 to 16, expressed as 12:16, and ratio of the number of boys to the number of girls is 16:12).

122

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[1] Expectations for unit rates in this grade are limited to non-complex fractions.
[3] While it is possible to define *ratio* so that A can be zero, this will rarely happen in context, and so the discussion proceeds assuming both A and B are non-zero.

123   Students work with models to develop their understanding of ratios. (**MP.2, MP.6**)

124   Initially students do not express ratios using fraction notation so that ratios can be

125   differentiated from fractions and from rates. Later, students understand that ratios can

126   be expressed in fraction notation, but that ratios are different from fractions in several

127   ways.

128

129   Ratios have associated *rates.* For example, in the ratio 3 cups of orange juice to 2 cups

130   of fizzy water, the *rate* is $\frac{3}{2}$ cups of orange juice per 1 cup of fizzy water. The term *unit*

131   *rate* refers to the numerical part of the rate; in the previous example, the unit rate is the

132   number $\frac{3}{2} = 1.5$. (The word "unit" is used to highlight the 1 in "per 1 unit of the second

133   quantity.") Students understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a:b$

134   (with $a, b \neq 0$), and use rate language in the context of a ratio relationship (**6.RP. 2▲**).

135

Examples of Ratio Language
1. If a recipe calls for a ratio of 3 cups of flour to 4 cups of sugar, then the *ratio* of flour to sugar is 3:4. This can also be expressed with units included as "3 cups flour: 4 cups sugar." The associated *rate* is "$\frac{3}{4}$ cups of flour per cup of sugar." The *unit rate* is the number $\frac{3}{4} = .75$.
2. If the soccer team paid $75 for 15 hamburgers, then this is a *ratio* of $75: 15 hamburgers or 75:15. The associated *rate* is $5 per hamburger. The *unit rate* is the number $\frac{75}{15} = 5$.

136

137   Students understand that rates always have units associated with them because they

138   result from dividing two quantities. Common unit rates are cost per item or distance per

139   time. In grade six, the expectation is that student work with unit rates is limited to

140   fractions in which both the numerator and denominator are whole numbers. Grade six

141   students use models and reasoning to find rates and unit rates.

142

Why must b not be equal to 0?
For a unit rate, or any rational number, $\frac{a}{b}$, the denominator b must not equal 0 since division by 0 is *undefined* in mathematics. To see that division by zero cannot be defined in a meaningful way, we related division to multiplication. That is, if $a \neq 0$ and if $\frac{a}{0} = x$ for some number x, then it must be true that $a = 0 \cdot x$. But since $0 \cdot x = 0$ for any x, there is no x that makes the equation $a = 0 \cdot x$ true. For a

> different reason, $\frac{0}{0}$ is undefined because it cannot be assigned a unique value. Indeed, if $\frac{0}{0} = x$, then $0 = 0 \cdot x$, which is true for *any* value of x. So what would x be?

143

144   Students understand ratios and their associated rates by building on their prior

145   knowledge of division concepts.

146

> **Example (MP.2, MP.6).** There are 2 brownies for 3 students. What is the amount of brownie that each student receives? What is the unit rate?
>
> **Solution:** This can be modeled to show that there is $\frac{2}{3}$ of a brownie for each student. The unit rate in this case is $\frac{2}{3}$. In the picture, each student is counted as they receive a portion of brownie, and it is clear that each student receives $\frac{2}{3}$ of a brownie.
>
> | 1 | 2 | 3 | | 1 | 2 | 3 |
> |---|---|---|---|---|---|---|

147

148   In general, students should be able to identify and describe any ratio using language

149   such as, "For every _____, there are _____."

150   (Adapted from Arizona 2012 and N. Carolina 2012)

151

Ratios and Proportional Relationships	6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

3.   Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
   a.   Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
   b.   Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*
   c.   Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
   d.   Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

152

153   Students make tables of equivalent ratios relating quantities with whole number

154   measurements, they find missing values in the tables, and plot the pairs of values on

155   the coordinate plane. They use tables to compare ratios. (**6.RP.3a▲**) Grade six

156   students work with tables of quantities in equivalent ratios (also called *ratio tables*) and

157   practice using ratio and rate language to deepen their understanding of what a ratio

158   describes. As students generate equivalent ratios and record ratios in tables, students

159   should notice the role of multiplication and division in how entries are related to each

160   other. Students also understand that equivalent ratios have the same unit rate. Tables

161   that are arranged vertically can help students see the multiplicative relationship between

162   equivalent ratios, and help avoid confusing ratios with fractions. (Adapted from The

163   University of Arizona Progressions Documents for the Common Core Math Standards

164   [Progressions] 6-7 Ratios and Proportional Relationships [RP] 2011).

165

Example: Representing Ratios in Different Ways. A juice recipe calls for 5 cups of grape juice for every 2 cups of peach juice. How many cups of grape juice are needed for a batch that uses 8 cups of peach juice?

Using Ratio Reasoning: (In the picture, 🍇 represents 1 cup of grape juice and 🍑 represents 1 cup of peach juice.) "For every 2 cups of peach juice there are 5 cups of grape juice, so I can draw groups of the mixture to figure out how much grape juice I would need."

"It's easy to see that when you have $4 \times 2 = 8$ cups of peach juice, you need $4 \times 5 = 20$ cups of grape juice."

Using a Table: "I can set up a table. That way it's easy to see that every time I add 2 more cups of peach juice, I need to add 5 cups of grape juice."

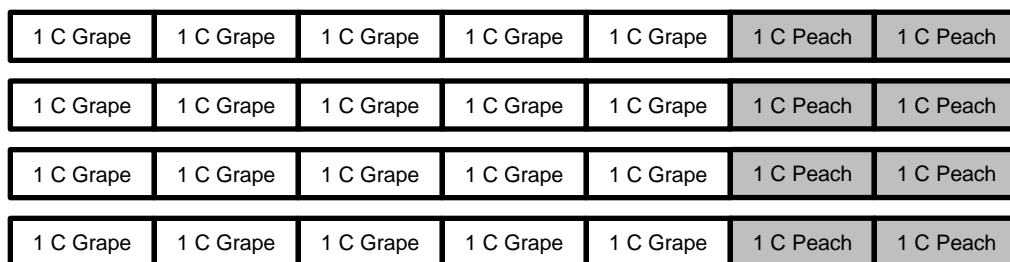| Cups of Grape Juice	Cups of Peach Juice
5	2
10	4
15	6
20	**8**
25	10

+5   +2

166

167   New to many sixth grade teachers are tape diagrams and double number line diagrams

168   (**6.RP.3**). A *tape diagram* expresses a ratio by representing parts with pieces of tape. It

169   is important to note that the units and size of the pieces of tape may not be evident
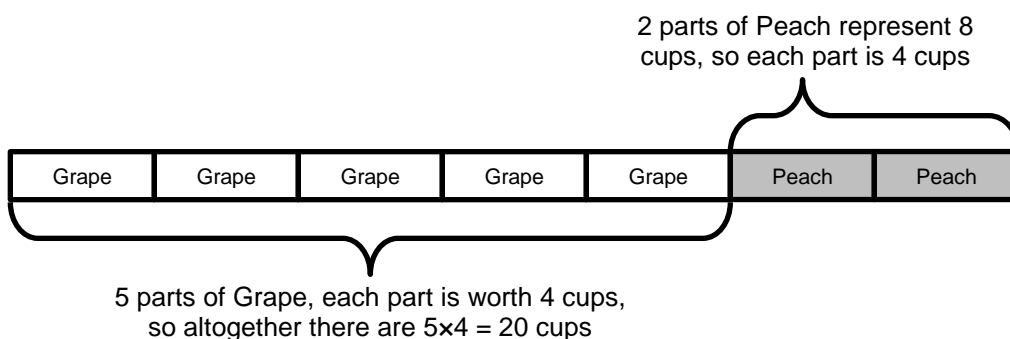
170 immediately in a given problem (but in a given problem each piece of tape has the same

171 size). Tape diagrams are often used in problems where the two quantities in the ratio

172 have the same units. A *double number line* diagram sets up two number lines with

173 zeroes connected. The same tick-marks are used on each line, but the number lines

174 have different units, which is central to how double number lines exhibit a ratio. The

175 following table shows how tape diagrams and double number lines can be used to solve

176 the previous example. (Adapted from Progressions on Ratio and Proportion 6-7.)

177

| Representing Ratios with Tape Diagrams and Double Number Line Diagrams. |
| --- |
| **Using a Tape Diagram (Beginning Method):** "I set up a tape diagram. I used pieces of tape to represent 1 cup of liquid, and then copied the diagram until I had 8 cups of peach juice."

 <table><tr><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Peach</td><td>1 C Peach</td></tr><tr><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Peach</td><td>1 C Peach</td></tr><tr><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Peach</td><td>1 C Peach</td></tr><tr><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Grape</td><td>1 C Peach</td><td>1 C Peach</td></tr></table> |
| **Using a Tape Diagram (Advanced Method):** "I set up a tape diagram in a ratio of 5:2. Since I know there should be 8 cups of peach juice, each piece of tape is worth 4 cups. That means there are 5 × 4 = 20 cups of grape juice."

 2 parts of Peach represent 8 cups, so each part is 4 cups

 <table><tr><td>Grape</td><td>Grape</td><td>Grape</td><td>Grape</td><td>Grape</td><td>Peach</td><td>Peach</td></tr></table>

 5 parts of Grape, each part is worth 4 cups, so altogether there are 5×4 = 20 cups |
| **Using a Double Number Line Diagram:** "I set up a double number line, with cups of grape juice on the top and cups of peach juice on the bottom. When I count up to 8 cups of peach juice, I see that this brings me to 20 cups of grape juice." |

Cups of grape juice

| 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 |

| 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

Cups of peach juice
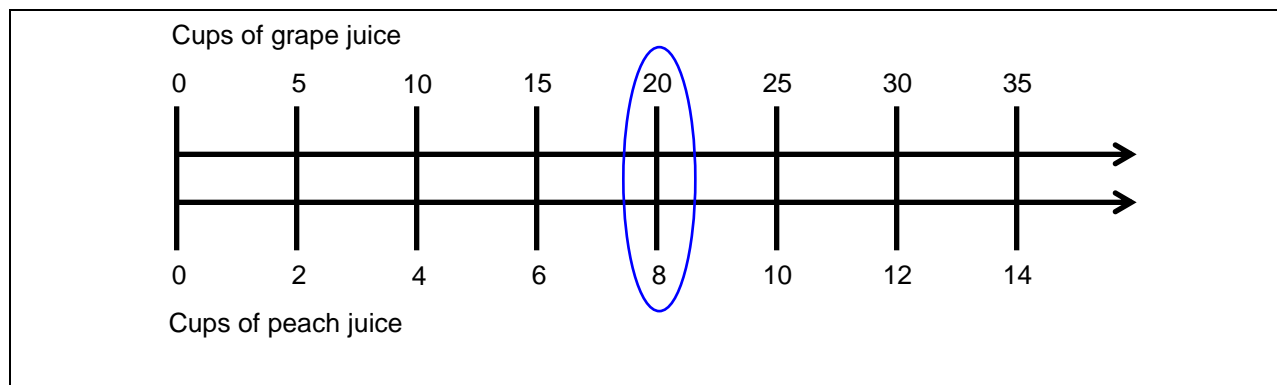
178

179 Representing ratios in various ways can help students see the additive and

180 multiplicative structure of ratios (**MP.7**). In standard (**6.RP.3.a▲**), students create

181 tables of equivalent ratios and represent the resulting data on a coordinate grid.

182 Eventually, students see this additive and multiplicative structure in the graphs of ratios,

183 which will be useful later when studying slopes and linear functions. (See also Standard

184 **6.EE.9▲.**)

185

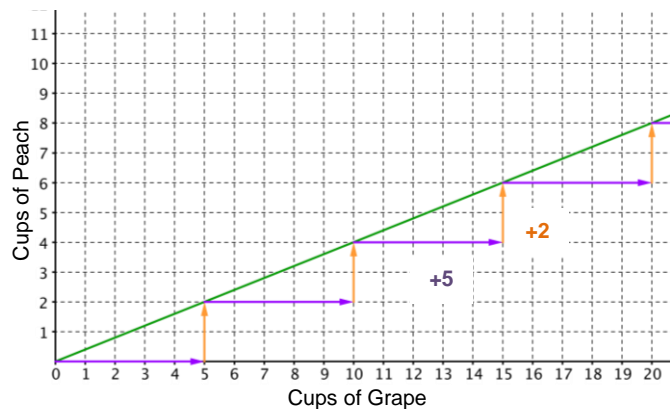| **Making Use of Structure in Tables and Graphs of Ratios.** |
|---|
| The additive and multiplicative structure of ratios can be pointed out to students in tables as well as graphs. (**6.RP.3.c**) |
| **Additive Structure:** |

Table:

Cups Grape	Cups Peach
5	2
10	4
15	6
20	8
25	10

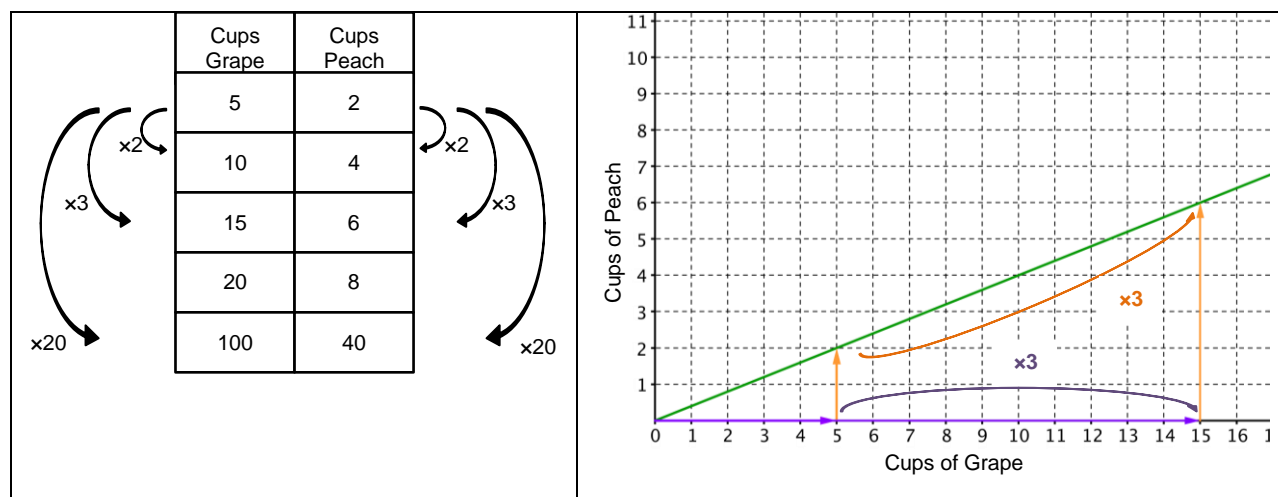+5 ... +2 (between each row)

Graph:

Multiplicative Structure:

Table: Graph:

186

187   (Adapted from Progressions 6-7 RP 2011)

188

189   As students solve similar problems they develop several mathematical practice

190   standards as they reason abstractly and quantitatively (**MP.2**), abstract information from

191   the problem, create a mathematical representation of the problem, and correctly work

192   with both part-part and part-whole situations. Students model with mathematics (**MP.4**)

193   as they solve the problem with tables and/or ratios. They attend to precision (**MP.6**) as

194   they properly use ratio notation, symbolism, and label quantities.

195

196   Following is a sample classroom activity that connects the Standards for Mathematical

197   Content and the Standards for Mathematical Practice, appropriate for students who

198   have already been introduced to ratios and associated rates.

199 **Connecting to the Standards for Mathematical Practice – Grade 6**
200

Standards	Explanations and Examples
6.RP.1: Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."* **6.RP.2**: Understand the concept of a unit rate *a/b* associated with a ratio *a:b* with *b* ≠ 0, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*1	**Sample Problem:** When Mr. Short is measured in paperclips, he is 6 paperclips tall. When he is measured in buttons, he is 4 buttons tall. Mr. Short has a daughter named Suzy Short. When Suzy Short is measured in buttons, she is 2 buttons tall. How many paperclips tall is Suzy Short? **Solution:** Since Mr. Short is both 6 paperclips tall and 4 buttons tall, it must be true that 1.5 paperclips is the same height as 1 button. Therefore, since Suzy Short is 2 buttons tall, she is 2×1.5=3 paperclips tall. Also, since Suzy Short is half the number of buttons tall as her father, she must be half the number of paperclips tall. **Classroom Connections:** The purpose of this problem is to introduce students to the notions of **ratio** and **unit rate**. Students can attempt to solve the problem and explain to another student how they got their answer. Diagrams should be encouraged if students have trouble beginning. The correct answer is arrived at, 3 paperclips, while the common wrong answer, 4 paperclips, is discussed. A wrong answer of 8 paperclips typically appears when students think *additively* as opposed to *multiplicatively*. A simple diagram shows that *for every 3 paperclips, there are 2 buttons,* and in this way the notion of ratio is introduced. The language of a *ratio of 3:2* can be introduced here. Pictures can also help illustrate the concept of an *associated rate*: that there are $\frac{3}{2} = 1.5$ *paperclips for every 1 button.* Follow up tasks might include: 1. Mr. Short's car is 15 paperclips long. How long is his car when measured in buttons? 2. Mr. Short's car is 7.5 paperclips wide. How wide is his car when measured in buttons? 3. Mr. Short's house is 12 buttons tall. How tall is his house when measured in paperclips? 4. Make a table that compares the number of buttons and number of paperclips. How does your table show the ratio of 3:2? **Connecting to the Standards for Mathematical Practice:** (MP.1) Students can be challenged to solve the problem with little background in ratios, and to try to discover a relationship between paperclips and buttons. Students make sense of the problem as they create a simple illustration or try to picture how buttons are related to paperclips. (MP.3) Students can be challenged to explain their reasoning for finding out how tall Suzy Short is in buttons. They can be asked to share with a partner or the whole class how they found their answer. (MP.6) Teachers can challenge students to use new vocabulary precisely when discussing their solution strategies. They are encouraged to explain why a ratio of 3:2 is equivalent to a unit ratio of 1.5:1. They include the units of paperclips and buttons in their solutions. (This problem is adapted from Lamon, Susan J. "Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers").

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201   In standard **6.RP.3b-d**, students apply their newfound ratio reasoning to various

202   situations in which ratios appear, including unit price, constant speed, percent, and the

203   conversion of measurement units. In sixth grade, generally only whole number ratios

204   are considered. The basic idea of *percent* is a particularly relevant and important topic

205   for young students to learn, as they will use percent throughout the rest of their lives

206   (**MP.4**). Percent will be discussed in a separate section that follows, but below are

207   several more examples of ratios and the reasoning expected in the **6.RP** domain.

Examples of Ratios and Ratio Reasoning Problems.	
1. On a bicycle, you can travel 20 miles in 4 hours. What distance can you travel in 1 hour? (**MP.2**)	**2.** At the pet store, a fish tank has guppies and goldfish in a ratio of 6:9. Show that this is the same as a ratio of 2:3.
Solution: Students might use a double number line diagram to represent the relationship between miles ridden and hours elapsed. They build on fraction reasoning from earlier grades to divide the double number line into 4 equal parts, and mark the DNL accordingly. It becomes clear that in 1 hour, one can ride 5 miles, for a rate of 5 miles per hour.	**Solution:** Students should be able to find equivalent ratios by drawing pictures or using ratio tables. Equivalent ratios will be closely linked to equivalent fractions later. A ratio of 6:9 might be represented in the following way, with black fish as guppies and white fish as goldfish: This picture can be rearranged to show 3 sets of 2 guppies and 3 sets of 3 goldfish, for a ratio of 2:3.
3. Use the information in the table given to find the number of yards that equals 24 feet.	**4.** The cost of 3 cans of pudding at *Superway Store* is $2.25 and the cost of 6 cans of the same pudding is $4.80 at *Giant Grocery*. Which store has the better price for this kind of pudding?
Solution: Students can solve this in several ways: 1. They can observe the associated rate from the table, 3 feet per yard, and they can use multiplication to see that 24 feet = 8 × 3 feet, so the answer is 8 × 1 yard = 8 yards. 2. They can notice that 24 feet = 4 × (6 feet), so that the answer is 4 × (2 yards) = 8 yards. 3. They can see that with ratios, you can *add* entries in a table because of the distributive	**Solution:** Students can solve this in several ways. 1. They can make a table of prices for different numbers of cans and compare the price for the same number of cans. 2. They can multiply the number of cans and their price at *Superway Store* by 2 to see that 6 cans there cost $4.50, so that one can buy the same number of cans at *Superway Store* for less than at *Giant Grocery* (6 for $4.80).

For problem 3, the table:

Feet	3	6	9	15	24
Yards	1	2	3	5	??

property: 9 feet + 15 feet = 24 feet 3(3 feet) + 5(3 feet) = 8 (3 feet) And since 3 feet = 1 yard, we have 8 yards.	3. Finally they can find the *unit price* at each store: $\frac{\$2.25}{3} = \$0.75$ per can at *Superway Store* $\frac{\$4.80}{6} = \$0.80$ per can at *Grocery Giant*.

208

209 Percent: A Special Type of Rate

210 In Standard **6.RP.3.c,** grade six students understand percent as a special type of rate,

211 and they use models and tables to solve percent problems. This is students' first formal

212 introduction to percent. Students understand percentages represent a rate per 100; for

213 example, to find 75% of a quantity means to multiply the quantity by $\frac{75}{100}$ or, equivalently,

214 by the fraction $\frac{3}{4}$. They come to understand this as they represent percent problems

215 with tables, tape diagrams, and double number line diagrams. Student understanding

216 of percent is related to their understanding of fractions and decimals. A thorough

217 understanding of place value helps students see the connection between decimals and

218 percent (for example, students understand that 0.30 represents $\frac{30}{100}$, which is the same

219 as 30%).

220

221 Students can use simple "benchmark percentages" (e.g., 1%, 10%, 25%, 50%, 75%, or

222 100%) as one strategy to solve percent problems (e.g., "what is 50% of a number"). By

223 reasoning about rates using the distributive property, students see that percentages can

224 be combined to find other percentages, and thus benchmark percentages become a

225 very useful tool when learning about percent. (**MP.5**)

Benchmark Percentages. (MP.7)
• 100% of a quantity is the entire quantity, or "1 times" the quantity.
• 50% of a quantity is half the quantity (since $50\% = \frac{50}{100} = \frac{1}{2}$), and 25% is one-quarter of a quantity (since $25\% = \frac{25}{100} = \frac{1}{4}$).
• 10% of a quantity is $\frac{1}{10}$ of the quantity (since $10\% = \frac{10}{100} = \frac{1}{10}$), so to find 10% of a quantity students can divide the quantity by 10. Similarly, 1% is $\frac{1}{100}$ of a quantity.
• 200% of a quantity is twice the quantity (since $200\% = \frac{200}{100} = 2$).

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- 75% of a quantity is $\frac{3}{4}$ of the quantity. Students also find that $75\% = 50\% + 25\%$, or $75\% = 3 \times 25\%$. Tape diagrams and DNLs can be useful for seeing this relationship.

226

227 A "percent bar" is a visual model, similar to a combined double number line and tape

228 diagram, which can be used to solve percent problems. Students can fold the bar to

229 represent benchmark percentages, such as 50% (half), 25% and 75% (quarters) and

230 10% (tenths). Teachers should connect percent to ratios, to point out to students that

231 percent is not an unrelated topic, but a useful application of ratios and rates.

Examples of Connecting Percent to Ratio Reasoning.

1. Andrew was given an allowance of $20. He used 75% of his allowance to go to the movies. How much money was spent at the movies?

Solution: "By setting up a percent bar, I can divide the $20 into four equal parts. I see that he spent $15 at the movies."

| $0 | $5 | $10 | $15 | $20 |

| 0% | 25% | 50% | 75% | 100% |
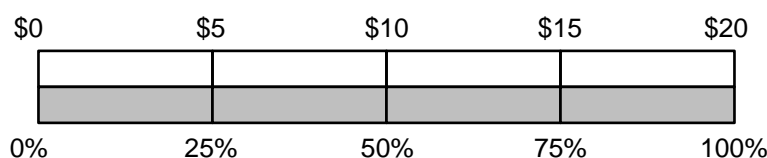
2. What percent is 12 out of 25?

Solutions: (a) "I set up a simple table and found that 12 out of 25 is the same as 24 out of 50, which is the same as 48 out of 100. So 12 out of 25 is 48%."

Part	12	24	48
Whole	25	50	100

(b) "I saw that 4 × 25 is 100, so I found 4 × 12 = 48. So 12 out of 25 is the same as 48 out of 100, or 48%."

(c) "I know that I can divide 12 by 25, since $\frac{12}{25} = 12 \div 25$. I got .48, which is the same as 48/100, or 48%."

232   (Adapted from Arizona 2012 and N. Carolina 2012)

233

234 There are several types of percent problems that students should become familiar with,

235 including finding the percentage represented by a part out of a whole, finding the

236 unknown part when given a percentage and whole, and finding an unknown whole when

237 a percentage and part are given. Students are not yet responsible for solving multistep

238 percent problems, such as finding sales tax, markups and discounts, or percent change.

239 The following examples illustrate these problem types, as well as how to use tables,

240 tape diagrams, and double number lines to solve them.

More Examples of Percent Problems.

Finding an Unknown Part: Last year Mr. Christian's class had 30 students. This year the number of students in his class is 150% of the number of students he had in his class last year. How many students does he have this year?

Solution: "Since 100% is 30 students, I know that 50% is 30 ÷ 2 = 15 students. This means that 150% is 3 × 15 = 45 students, since 150% = 3 × 50%. His class has 45 students in it this year."

Finding an Unknown Percentage: When the entire sixth grade of 240 students was polled, results showed that 96 students were dissatisfied with the music at the dance. What percentage of the sixth grade does this represent?

Solution: "I set up a double number line (DNL) diagram. It was easy to find 50% was 120 students. This meant that 10% was 120 ÷ 5 = 24 students. I noticed that 96 ÷ 24 is 4. Reading my DNL, this means that 40% of the students were dissatisfied (40% = 4 × 10%)."

Number of Students
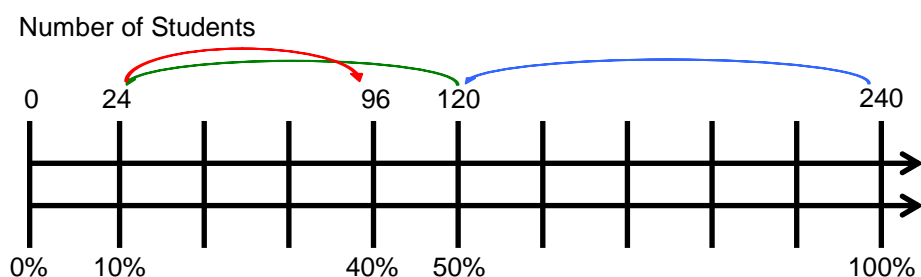
Finding an Unknown Whole: If 75% of the budget is $1,200, what is the full budget?

Solution: By setting up a fraction bar, I can find 25% since I know 75%. Then, I multiply by 4 to give me 100%. Since 25% is $400, I see that 100% is $1,600."
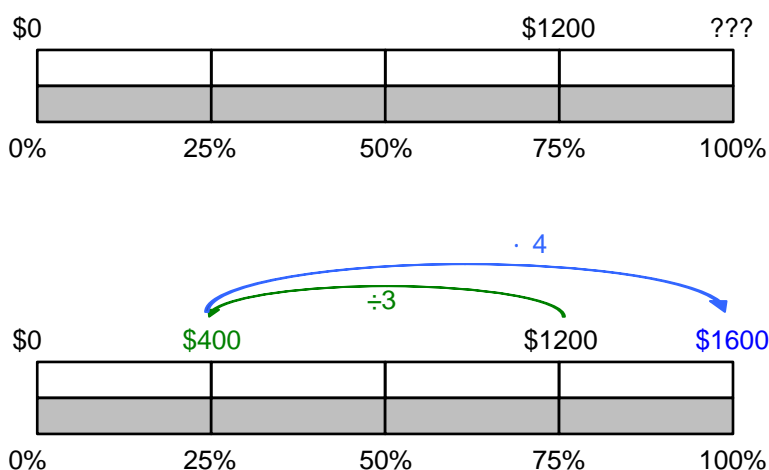
In problems such as this one, teachers can use scaffolding questions, such as:

- If you know 75% of the budget, how can we determine 25% of the budget?
- If you know 25% of the budget, then how can this help you find 100% of the budget?

241   (Progressions 6-7 RP 2011)

242

243   When students have had sufficient practice solving percent problems using tables and

244   diagrams, they can be led to representing percentages as decimals as a way to solve

245   problems. For instance the previous three problems can be solved using methods such

246   as those shown below.

247

If the class has 30 students, then 150% can be found by finding the fraction $\frac{150}{100} = \frac{15}{10} = 1.5$ times 30. The answer is 1.5 × 30 = 45.
Since 96 out of 240 were dissatisfied with the music at the dance, this means that $\frac{96}{240} = .4 = \frac{40}{100} = 40\%$ were dissatisfied with the music.
Since the budget is unknown, let's call it B. Then we know that 75% of the budget is $1200, which means that $0.75B = 1200$. This can be solved by finding $B = 1200 \div .75$. Alternately, students may see that $$\frac{75}{100} = \frac{1200}{B} \text{ which can be rewritten as } \frac{3}{4} = \frac{1200}{B}$$ By reasoning with equivalent fractions, since $\frac{3}{4} = \frac{3 \times 400}{4 \times 400} = \frac{1200}{1600}$, we see that $B = \$1600$.

248

249   In reasoning about and solving percent problems, students develop mathematical

250   practices as they use a variety of strategies to solve problems, use tables and diagrams

251   to represent problems (**MP.4**), and reason about percent (**MP.1, MP.2**).

252

Common Student Misconceptions with Ratios and Percent.
• While ratios can be represented as fractions, the connection is subtle. Fractions express a part-to-whole comparison, but ratios can express part-to-whole or part-to-part comparisons. Care should be taken if teachers choose to represent ratios as fractions at this grade level. • Proportional situations can have several ratios associated with them. For instance, in a 1 part juice to 2 parts water mixture, there is a ratio of 1 part juice to 3 *total parts* (1:3), as well as the more obvious ratio of 1:2. • Students must carefully reason why they can add ratios. For instance, in a mixture with lemon drink and fizzy water in a ratio of 2:3, mixtures made with ratios 2:3 and 4:6 can be added to give a mixture of ratio 6:9, equivalent to 2:3. This is because, $$2(\text{parts lemon}) + 4(\text{parts lemon}) = 6(\text{parts lemon})$$ $$3(\text{parts fizzy}) + 6(\text{parts fizzy}) = 9(\text{parts fizzy})$$ However, one would *never* add fractions by adding numerators and denominators:

253

$$\frac{2}{3} + \frac{4}{6} \neq \frac{6}{9}.$$

254 A detailed discussion of ratios and proportional relationships is provided online at Draft

255 6–7 Progression on Ratios and Proportional Relationships (Progressions 6-7 RP 2011).

256

257   **Domain: The Number System**

258

259 In grade six students complete their understanding of division of fractions and extend

260 the notion of number to the system of rational numbers, which includes negative

261 numbers. Students also work toward fluency with multi-digit division and multi-digit

262 decimal operations.

263

The Number System	6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?*

264

265 In grade five students learned to divide whole numbers by unit fractions and unit

266 fractions by whole numbers. These experiences lay the conceptual foundation for

267 understanding general division of fractions in sixth grade. Grade six students continue

268 to develop division by using visual models and equations to divide fractions by fractions

269 to solve word problems (**6.NS.1▲**). Student understanding of the meaning of the

270 operations with fractions builds upon the familiar understandings of these meanings with

271 whole numbers and can be supported with visual representations. To help students

272 make this connection, teachers might have students think about a simpler problem with

273 whole numbers and then use the same operation to solve with fractions.

274

275   Looking at the problem through the lens of "How many groups?" or "How many in each

276   group?" helps students visualize what is being sought. Encourage students to explain

277   their thinking and to recognize division in two different situations—measurement

278   division, which requires finding how many groups (e.g., how many groups can you

279   make?) and fair-share division, which requires equal sharing (e.g., finding how many

280   are in each group). In fifth grade, students represented division problems like $4 \div \frac{1}{2}$ with

281   diagrams and reasoned why the answer is 8 (e.g. how many halves are in 4?). They

282   may have discovered that $4 \div \frac{1}{2}$ can be found by multiplying 4×2 (i.e., each whole gives

283   2 halves, so there are 8 halves altogether). Similarly, students may have found that

284    $\frac{1}{3} \div 5 = \frac{1}{3} \times \frac{1}{5}$. These generalizations will be exploited when developing general methods

285   for dividing fractions. Teachers should be aware that making visual models for general

286   division of fractions can be difficult; it can be simpler to move to discussing general

287   methods of dividing fractions and use one of these methods to solve problems.

288

289   The following examples illustrate how reasoning about division can help students

290   understand fraction division, before moving to general methods.

291

Some Examples of Division Reasoning with Fractions.

1. Three people share $\frac{2}{3}$ of a pound of chocolate. How much chocolate does each person get?

Solution: This problem can be represented by $\frac{2}{3} \div 3$. To solve it, students might represent the chocolate

with a diagram such as the one below. There are two $\frac{1}{3}$-pound pieces represented in the picture.

Students can see that $\frac{1}{3}$ divided among three people is $\frac{1}{9}$. Since there are 2 such pieces, each person

receives $\frac{2}{9}$.

| $\frac{1}{3}$ | $\frac{1}{3}$ | | → | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | | | |
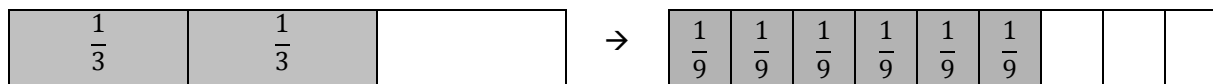
Problems like this one can be used to support the fact that, in general, $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$.

2. Manny has $\frac{1}{2}$ yard of fabric to make book covers. Each book cover is made from $\frac{1}{8}$ yard of fabric. How

many book covers can Manny make?

Solution: Students can think, "How many $\frac{1}{8}$-yard pieces can I make from $\frac{1}{2}$-yard of fabric?" By subdividing

---

the $\frac{1}{2}$-yard of fabric into eighths (of a yard), students can see that there are 4 such pieces.
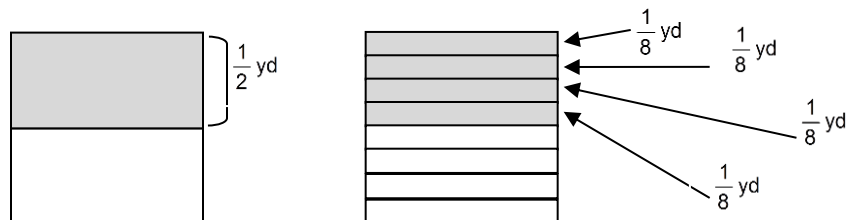
Problems like this one can be used to support the fact that, in general, $\frac{1}{b} \div \frac{1}{d} = \frac{1}{b} \times d$.

3. You are making a recipe that calls for $\frac{2}{3}$ cup of yogurt. You have $\frac{1}{2}$ cup of yogurt from a snack pack. How much of the recipe can you make?

Solutions: Students can think,

"how many portions of size $\frac{2}{3}$-cup

can be made from only $\frac{1}{2}$ cup?"

Students can reason that the answer will be less than 1, as there is not even enough yogurt to make 1 full recipe. What is difficult about

$\frac{2}{3}$ cup, subdivided into sixths

$\frac{1}{2}$ cup, represents 3 of those 4 sixths

this problem is that it is not immediately apparent how to find thirds from halves. Students can convert the fractions into ones with common denominators to make the problem more accessible. Since $\frac{2}{3} = \frac{4}{6}$ and $\frac{1}{2} = \frac{3}{6}$, it makes sense to represent the $\frac{2}{3}$-cup required for the recipe divided into $\frac{1}{6}$-cup portions. As the diagram shows, the recipe calls for $\frac{4}{6}$-cup, but we only have 3 of those 4 sixths needed. Each sixth is $\frac{1}{4}$ of a recipe, and we have 3 of them, so we can make $\frac{3}{4}$ of a recipe.
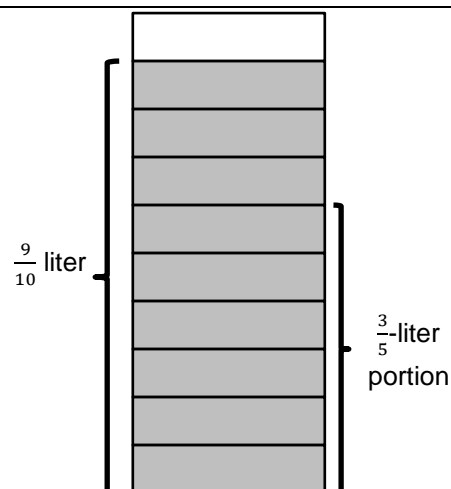
Problems like this one can be used to support the division by common denominators strategy.
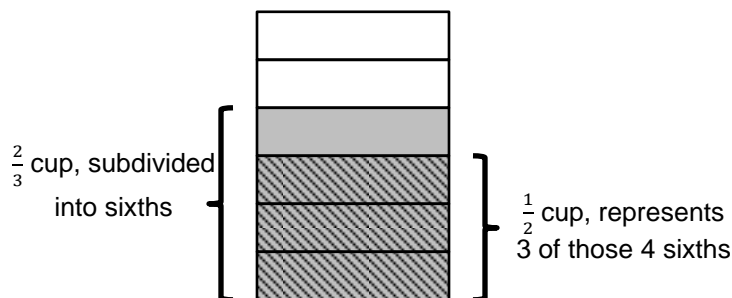
4. A certain type of water bottle holds $\frac{3}{5}$ of a liter of liquid. How many of these bottles could be filled with $\frac{9}{10}$ of a liter of juice?

Solution: The picture shows $\frac{9}{10}$ of a liter of juice. Since 6 tenths make $\frac{3}{5}$ of a liter, clearly one bottle can be filled. The remaining $\frac{3}{10}$ of a liter represents $\frac{1}{2}$ of a bottle, so it makes sense to say that

$1\frac{1}{2}$ bottles could be filled.

Notice that $\frac{1}{10} \div \frac{1}{5} = \frac{1}{2}$, meaning that there is one-half of $\frac{1}{5}$ in each

$\frac{1}{10}$. This means that in 9 tenths, there are 9 halves of $\frac{1}{5}$. But

$\frac{9}{10}$ liter

$\frac{3}{5}$-liter portion

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since the capacity of a bottle is *three* of these fifths, there are $\frac{9}{2} \div 3 = \frac{3}{2}$ of these bottles. This line of reasoning supports the idea that one can divide numerators and denominators, i.e. $\frac{9}{10} \div \frac{3}{5} = \frac{9 \div 3}{10 \div 5} = \frac{3}{2}$.

292   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

293

Common Misconception: Students may confuse dividing a quantity **by** $\frac{1}{2}$ with dividing a quantity **in half**. Dividing **by** $\frac{1}{2}$ is finding how many $\frac{1}{2}$-sized portions there are, as in "dividing 7 **by** $\frac{1}{2}$," which is $7 \div \frac{1}{2} = 14$. On the other hand, to divide a quantity **in half** is to divide the quantity into two parts equally, as in "dividing 7 **in half**" yields $\frac{7}{2} = 3.5$. Students should understand that dividing **in half** is the same as dividing by 2. (Adapted from KATM 6th FlipBook 2012)

294

295   Students should also connect division of fractions with multiplication, for example, in the

296   problems above, students should reason that it makes sense that $\frac{2}{3} \div 3 = \frac{2}{9}$, since

297    $3 \times \frac{2}{9} = \frac{2}{3}$. Also, it makes sense that $\frac{1}{2} \div \frac{1}{8} = 4$ since $\frac{1}{8} \times 4 = \frac{1}{2}$, and that $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$ since

298    $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$. All of these relationships can be made sense of using the previous diagrams.

299   The relationship between division and multiplication will be used to develop general

300   methods for dividing fractions, explained below.

301

302   **General Methods for Dividing Fractions.**

303   1. *Finding common denominators*: Interpreting division as measurement division

304   allows one to divide fractions by finding common denominators (i.e. common

305   *denominations*). For example, to divide $\frac{7}{8} \div \frac{2}{5}$, we find a common denominator, so

306   we rewrite $\frac{7}{8}$ as $\frac{35}{40}$ and $\frac{2}{5}$ as $\frac{16}{40}$. Now the problem becomes, "how many groups of

307   16-fortieths can we get out of 35-fortieths?" That is, the problem becomes

308           $35 \div 16 = \frac{35}{16}$. This approach of finding common denominators reinforces the

309   linguistic connection between "denominator" and "denomination."

310   2. *Dividing numerators and denominators (special case)*: By thinking about the

311   relationship between division and multiplication, students can reason that a

312   problem like $\frac{8}{15} \div \frac{2}{5} = ?$ is the same as finding $\frac{2}{5} \times ? = \frac{8}{15}$. Students can see that the

313   fraction $\frac{4}{3}$ represents the missing factor, but this is the same result as if one

314   simply divided numerators and denominators: $\frac{8}{15} \div \frac{2}{5} = \frac{8 \div 2}{15 \div 5} = \frac{4}{3}$. While this

315   strategy works in general, it is particularly useful when the numerator and

316   denominator of the divisor are factors of the numerator and denominator of the

317   dividend, respectively.

318   3. *Dividing numerators and denominators (leading to general case)*: By rewriting

319   fractions as equivalent fractions, students can use the previous strategy in other

320   cases, for instance when the denominator of the divisor is not a factor of the

321   denominator of the dividend. For example, when finding $\frac{2}{3} \div \frac{2}{7}$, students can

322   rewrite $\frac{2}{3}$ as $\frac{14}{21} = \frac{2 \times 7}{3 \times 7}$, to yield

$$\frac{2}{3} \div \frac{2}{7} = \frac{14}{21} \div \frac{2}{7} = \frac{14 \div 2}{21 \div 7} = \frac{7}{3}.$$

323   4. *Dividing numerators and denominators (general case):* When neither the

324   numerator nor denominator of the divisor is a factor of those of the dividend, we

325   can again use equivalent fractions to develop a strategy. For instance, with a

326   problem like $\frac{3}{4} \div \frac{5}{7}$, we can rewrite the fraction $\frac{3}{4}$ as $\frac{3 \times 5 \times 7}{4 \times 5 \times 7}$ and do the division. In

327   fact, when we leave the fraction in this form we see that:

$$\frac{3}{4} \div \frac{5}{7} = \frac{3 \times 5 \times 7}{4 \times 5 \times 7} \div \frac{5}{7} = \frac{(3 \times 5 \times 7) \div 5}{(4 \times 5 \times 7) \div 7} = \frac{3 \times 7}{4 \times 5} = \frac{3}{4} \times \frac{7}{5}.$$

328   This line of reasoning shows why it makes sense to invert the divisor and multiply

329   to find the result.

330   Teaching the "invert and multiply" model for dividing fractions without developing an

331   understanding of why it works can confuse students and interfere with their ability to

332   apply division of fractions to solve word problems. Teachers can gradually develop

333   strategies such as those above, so that students see they can generally divide fractions

334   in two ways:

335   • Divide the first fraction (dividend) by the top and bottom numbers (numerator and

336   denominator) of the second fraction (divisor).

337   • Flip the second fraction (divisor) and multiply the first (dividend) by it.

338

339

340   [Note: Sidebar]

The following is an algebraic argument that $\frac{a}{b} \div \frac{c}{d} = x$ precisely when $x = \frac{a}{b} \times \frac{d}{c}$. Starting with $\frac{a}{b} = x \cdot \frac{c}{d}$, we argue that if we multiply both sides of the equation by the multiplicative inverse of $\frac{c}{d}$ we can isolate x on the right. Thus we examine $\frac{a}{b} \cdot \frac{d}{c} = \left(x \cdot \frac{c}{d} \right) \cdot \frac{d}{c}$. Continuing the computation on the right, we see that $\left(x \cdot \frac{c}{d} \right) \cdot \frac{d}{c} = x \cdot \left(\frac{c}{d} \cdot \frac{d}{c} \right) = x \cdot 1 = x$. Since $\frac{a}{b} \div \frac{c}{d} = x$ as well, we have $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

341

The Number System   6.NS

Compute fluently with multi-digit numbers and find common factors and multiples.
2. Fluently divide multi-digit numbers using the standard algorithm.
3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.
4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*

342

343   In previous grades, students built a conceptual understanding of operations with whole

344   numbers and became fluent in multi-digit addition, subtraction, and multiplication. In

345   grade six, students work toward fluency with multi-digit division and multi-digit decimal

346   operations (**6.NS.2-3**). Fluency with the standard algorithms is expected, but an

347   algorithm is defined by its steps and not the way those steps are recorded in writing, so

348   minor variations in written methods are acceptable.

349   [Note: Sidebar]

FLUENCY

In kindergarten through grade six there are individual content standards that set expectations for fluency with computations using the standard algorithm (e.g., "fluently" divide multi-digit numbers (**6.NS.2**) and "fluently" add, subtract, multiply, and divide multi-digit decimals (**6.NS.3**) using the standard algorithm). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding (such as reasoning about quantities, the base-ten system, and properties of operations), thoughtful practice, and extra support where necessary.

The word "fluent" is used in the standards to mean "reasonably fast and accurate" and the ability to use

certain facts and procedures with enough facility that using them does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency for single-digit numbers in each grade can involve a mixture of just knowing some answers, knowing some answers from patterns, and knowing some answers from the use of strategies.

350

351   [Note: Sidebar]

Focus, Coherence, and Rigor

Division was introduced in grade three conceptually, as the inverse of multiplication. In grade four, students continued using place-value strategies, properties of operations, and the relationship with multiplication, area models, and rectangular arrays to solve problems with one-digit divisors and develop and explain written methods. This work was extended to include two-digit divisors and all operations with decimals to hundredths in grade five. In grade six, fluency with the algorithms for division is reached **(6.NS.2)**.

352

353   Grade six students fluently divide using the standards algorithm (**6.NS.2**). Students

354   should examine several methods to record division of multi-digit numbers and focus on

355   a variation of the standard algorithm that is efficient and that makes sense to them.

356   They can compare variations to understand how the same step can be written

357   differently but still have the same place value meaning. All such discussions should

358   include place value terms. Students should see examples of standard algorithm

359   division that can be easily connected to place value meanings.

360

Example. *Scaffold division* is a variation of the standard algorithm in which partial quotients are written to the right of the division steps rather than above.

To find the quotient 3440 ÷ 16, students can begin by asking, "How many groups of 16 are in 3440?" This is a measurement interpretation of division, and can form the basis of the standard algorithm. Students estimate that there are at least 200 groups of 16, since 2 × 16 = 32 and therefore 200 × 16 = 3200. Continuing, they see that they would then ask, "How many groups of 16 are in what remains, 240 (3440−3200)?" Clearly, there are at least 10. The next remainder is then 80 = 240−160, and we see that there are 5 more groups of 16 in this remaining 80. The quotient in this strategy is then found to be

Divisor	Dividend	Quotient
16	3440 −3200	200
	240 − 160	10
	80 − 80	5
	0	**215**
	Remainder	**Quotient**

> 200+10+5 = 215.

361

362   The partial quotients can also be written above each other over the dividend (see later

363   example). Students can also consider writing single digits instead of totals, provided

364   they can explain why they do so with place value reasoning, dropping all of the 0s in the

365   quotients and subtractions in the dividend; they then write 215 step-by-step above the

366   dividend (see later example). In both cases students use place value reasoning.

367

Example. Division using single digits instead of totals.

If writing only single digits, being attentive to place value language, teachers can ask, "how many groups of 16 are in 34 (hundreds)?" Since there are two groups of 16 in 34, there are 2 (hundred) groups of 16 in 34 (hundreds), so we record this with a 2 in the hundreds place above the dividend. The product of 2 and 16 is recorded, and we subtract 32 from 34, understanding that we are subtracting 32 hundreds from 34 hundreds, yielding 2 hundreds remaining. Next, when we "bring the 4 down to write 24," we understand this as moving to the digit in the dividend necessary to obtain a number larger than the divisor. Again, we focus on the fact that there are 24 (tens) remaining, and so the question becomes, "How many groups of 16 are in 24 tens?" The algorithm continues and the quotient is found.

368

369   Students should have experience with many examples such as the previous one and

370   the one that follows. Teachers should be prepared to support discussions involving

371   place value should misunderstanding arise. There may be other effective ways to

372   include place value concepts in explaining a variation of the standard algorithm for

373   division, and teachers are encouraged to find a method that works for them and their

374   students. Overall, teachers should remember that the standards support coherence of

375   learning and conceptual understanding, and instruction that builds on students' previous

376   mathematical experiences is crucial. (Refer to the Grade Five chapter for more

377   explanation of division strategies; see also the example below.)

378

Connecting Place Value and Division Algorithms			
Algorithm 1	Explanation	Algorithm 2	Explanation
200 32 ⟌ 8456	There are 200 groups of 32 in 8456	2 32 ⟌ 8456	There are 2 (hundred) groups of 32 in 84 (hundred)
200 32 ⟌ 8456	200 times 32 is 6400, so we subtract and	2 32 ⟌ 8456	2 times 32 is 64, so there are 64

− 6400 2056	find there is 2056 left to divide	− 64 205	(hundreds) to subtract from 84 (hundreds). We include the 5 with what is leftover since the dividend (205) must be larger than the divisor.
60 200 32)‾8456 − 6400 2056	There are 60 groups of 32 in 2056	26 32)‾8456 − 64 205	Now, we see there are 6 (tens) groups of 32 in 205 (tens)
60 200 32)‾8456 − 6400 2056 − 1920 136	60 times 32 is 1920, so we subtract and find there is 136 left to divide	26 32)‾8456 − 64 205 − 192 136	6 times 32 is 192, so there are 192 (tens) to subtract from 205 (tens). We include the 6 with what is leftover.
4 60 200 32)‾8456 − 6400 2056 − 1920 136	There are 4 groups of 32 in 136	264 32)‾8456 − 64 205 − 192 136	Now, we see there are 4 groups of 32 in 136.
4 60 200 32)‾8456 − 6400 2056 − 1920 136 − 128 8	4 times 32 is 128, so we subtract and find there is 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is 200+60+4 = 264 with a remainder of 8, or $264\frac{8}{32} = 264\frac{1}{4}$. Another way to say this is $8456 = 32(264) + 8$.	264 32)‾8456 − 64 205 − 192 136 − 128 8	4 times 32 is 128, so we subtract and find there is 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is 200+60+4 = 264 with a remainder of 8, or $264\frac{8}{32} = 264\frac{1}{4}$. Another way to say this is $8456 = 32(264) + 8$.

(Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

Standard (**6.NS.3**) requires sixth grade students to fluently apply standard algorithms in

working with operations with decimals. In grades four and five students learned to add,

subtract, multiply, and divide decimals (to hundredths) with concrete models, drawings,

and strategies and used place value to explain written methods for these operations. In

385   grade six students become fluent in the use of some written variation of the standard

386   algorithms of each of these operations.

387

388   The notation for decimals (numbers written using places to the right of the ones place)

389   depends upon the regularity of the place value system across all places to the left and

390   right of the ones place. This understanding explains why addition and subtraction of

391   decimals can be accomplished with the same algorithms as with for whole numbers;

392   "like" values or units (such as tens or thousandths) are combined. Students should

393   have opportunities to solve problems that include zeros and problems in which they

394   might add zeros to be sure that they add or subtract like places. When adding and

395   subtracting decimals, a conceptual approach might instruct students to line up place

396   values, as opposed to "lining up the decimal point", in order to support consistent

397   student understanding of place value ideas.

398

Focus, Coherence, and Rigor.

Students should discuss how addition and subtraction of all quantities have the same basis: They add or subtract like place-value units (whole numbers and decimal numbers), add or subtract like unit fractions, or add or subtract like measures. Thus, addition and subtraction are consistent concepts across grade levels and number systems.

399

400   In grade five students multiplied decimals to hundredths. They understood that

401   multiplying by decimals moves the decimal point as many places to the right as there

402   are places in the multiplying decimal (see Grade Five chapter narrative for standards

403   **5.NBT.1-2,** starting on page 10). In grade six students extend and apply their place

404   value understanding to fluently multiply multi-digit decimals (**6.NS.3)**. Writing decimals

405   as fractions whose denominator is a power of 10 can be used to explain the "decimal

406   point rule" in multiplication. For example,

$$2.4 \times 0.37 = \frac{24}{10} \times \frac{37}{100} = \frac{24 \times 37}{10 \times 100} = \frac{888}{1000} = 0.888.$$

407   This logical reasoning based on place value and decimal fractions justifies the typical

408   rule, "count the decimal places in the numbers and insert the decimal point to make that

409   many places in the product."

410

411   The general methods used for computing quotients of whole numbers extend to

412   decimals with the additional concern of where to place the decimal point in the quotient.

413   Students have experienced dividing decimals to hundredths in grade five, but in grade

414   six they move to using standard algorithms for doing so. In simpler cases, like with 16.8

415   ÷ 8, students can simply apply the typical division algorithm, paying mind to place value.

416   When problems get more difficult, e.g., when the divisor also has a decimal point, then

417   students may need to use strategies involving rewriting the problem through changing

418   place values. Reasoning similar to that for multiplication can be used to explain the rule

419   that "when the decimal point in the divisor is moved to make a whole number, the

420   decimal point in the dividend should be moved the same number of places." For

421   example, for a problem like 4.2 ÷ .35, wherein someone might give a rote recipe: "move

422   the decimal point two places to the right" in .35 and also in 4.2, teachers can instead

423   appeal to the idea that one can make a simpler but equivalent division problem by

424   multiplying both numbers by 100 and still obtain the same quotient. That is,

$$4.2 \div .35 = (4.2 \times 100) \div (.35 \times 100) = 420 \div 35 = 12.$$

425   Attention to student understanding of place value is of the utmost importance. There is

426   no conceptual understanding gained by referring to this only as "moving the decimal

427   point," teachers can refer to this more meaningfully as "multiplying by $\frac{n}{n}$ in the form of

428   $\frac{100}{100}$."

429

Examples of Decimal Operations.
1. Maria had 3 kilograms of sand for a science experiment. She had to measure out exactly 1.625 kilograms for a sample. How much sand will be left after she measures out the sample? **Solution:** Student thinks, "I know that 1.625 is a little more than 1.5, so I should have about 1.5 kilograms remaining. I need to subtract common place values from each other, and I notice that 1.625 has three place values to the right of the ones place, so if I make 0s in the tenths, hundredths and thousandths places of 3 to make 3.000, then the numbers have the same number of place values. Then it's easier to subtract: $3.000 - 1.625 = 1.375$. There are 1.375 kilograms left."
2. How many ribbons 1.5 meters long can Victor cut from a cloth that is 15.75 meters long? **Solution:** Student thinks, "This looks like a division problem, and since I can multiply both numbers by the same amount and get the same answer, I'll just multiply both numbers by 100. So now I need to find

> 1575 ÷ 150 and this will give me the same answer. I did the division and I got 10.5, which means that Victor can make 10 full ribbons and he has enough leftover to make half a ribbon."

430

431   In fourth grade students identified prime numbers, composite numbers, and factor pairs.

432   In sixth grade students build on prior knowledge and find the greatest common factor

433   (GCF) of two whole numbers less than or equal to 100 and find the least common

434   multiple (LCM) of two whole numbers less than or equal to 12 (**6.NS.4**). Teachers might

435   employ compact methods for finding the LCM and GCF of two numbers, such as the

436   ladder method illustrated below, among other methods (such as listing multiples of each

437   number and identifying the least they have in common for LCM, etc.).

438

Example: Ladder Method for Finding GCF and LCM		
To find the LCM and GCF of 120 and 48, one can use the "ladder method," which systematically finds common factors of 120 and 48, and leaves us with the factors that 120 and 48 do not have in common. The GCF becomes the product of all those factors that 120 and 48 share, while the LCM is the product of the GCF and the remaining uncommon factors of 120 and 48. With the ladder method, common factors (3, 4, 2 in this case) are divided from the starting and remaining numbers until no more common factors to divide (5, 2). The GCF is then $3 \cdot 4 \cdot 2 = 24$ while the LCM is $24 \cdot 5 \cdot 2 = 240$.	**Common Factors**	**Remaining Numbers**
	3	120	48
	4	40	16
	2	10	4
		5	2

439

The Number System   6.NS

Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

    a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., –(–3) = 3, and that 0 is its own opposite.

    b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the

> coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.
>
> c.   Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

440

441   In grade six students begin the formal study of negative numbers, their relationship to

442   positive numbers, and the meaning and uses of absolute value. Students use rational

443   numbers (expressed as fractions, decimals, and integers) to represent real-world

444   contexts and they understand the meaning of 0 in each situation (**6.NS.5▲**). Count

445   models (positive/negative electric charge, credits/debits), where the context gives rise to

446   the basic meaning of $0 = (+n) + (-n)$, will help develop student understanding of the

447   relationship between a number and its opposite. In addition, measurement contexts

448   such as temperature and elevation can contribute to student understanding of these

449   ideas. (**MP.1, MP.2, MP.4**)

450

451   Note that the standards do not specifically mention the set of the *integers* (which

452   consists of the whole numbers and their opposites) as a distinct set of numbers.

453   Rather, the standards are focused on student understanding of the set of *rational*

454   *numbers* in general (which consists of whole numbers, fractions, and their opposites).

455   Thus, while early instruction in positives and negatives will likely start with examining

456   whole numbers and their opposites, students must also experience working with

457   negative fractions (and decimals) at this grade level. Ultimately, students learn that all

458   numbers have an "opposite."

459

Examples of Rational Numbers in Context.

1.   All substances are made up of atoms, and atoms have protons and electrons. A proton has a "positive charge", represented by "+1," and an electron has a "negative charge," represented by "−1." A group of 5 protons has a total charge of +5 and a group of 8 electrons has a total charge of −8. One positive charge combines with one negative charge to result in a "neutral charge", which we can represent by (+1) + (−1) = 0. So for example, a group of 4 protons and 4 electrons together would have a neutral charge since there are 4 positive charges to combine with 4 negative charges. We

---

could write this as: (+4) + (−4) = 0.

    a. What is the overall charge of a group of 3 protons and 3 electrons?

    b. What is the overall charge of a group of 5 protons with no electrons?

    c. What is the overall charge of a group of 4 electrons with no protons?

2. In a checking account, "credits" to the account are recorded as positive numbers (since they are adding money to the account), and "debits" to the account are recorded as negative numbers (since they are taking away money from the account).

    a. Explain the meaning of an account statement that reads a total balance of −$100.15.

    b. Explain the meaning of an account statement that reads a total balance of $225.78.

    c. If someone's bank statement reads −$45.67, then explain how they can get to a $0 balance.

3. At any place on Earth, the "elevation" of the ground you are standing on is how far above or below the average level of water in the ocean (called "Sea Level") the ground is.

    a. Discuss with your neighbor what an elevation of 0 means. Sketch a picture of what you think this means.

    b. Death Valley's Badwater Basin, located in California, is the point of lowest elevation in North America, at 282 feet below sea level. Explain why we would use a negative rational number to express this elevation.

    c. Mount Whitney is the highest mountain in California, at a height of 14,505 feet above sea level. Explain why we would use a positive number to express this elevation.

460

461 In prior grades students worked with positive fractions, decimals, and whole numbers on

462 the number line and in the first quadrant of the coordinate plane. In sixth grade,

463 students extend the number line to represent all rational numbers focusing on the

464 relationship between a number and its opposite, namely that they are equidistant from 0

465 on a number line. (**6.NS.6▲**) Number lines may be either horizontal or vertical (such as

466 on a thermometer); experiencing both will facilitate students' movement from number

467 lines to coordinate grids.

468

469   [Note: Sidebar]

The "Minus Sign."

The minus sign (−) has several uses in mathematics. Since Kindergarten, students have used this symbol to represent subtraction. Now, they are responsible for understanding that the same symbol can

be used to mean **negative**, as in -5. (Negative numbers have also been represented with a "raised" minus sign, such as in $^-5$; however, this practice is not consistent and so teachers should simply use the more common minus sign.) However, students must also learn that the minus sign represents **the opposite of**, as in, "-5 is the opposite of 5 since they are both the same distance from 0." This latter use is probably the most important, as it can be applied to cases such as, "$-(-9)$ is the opposite of the opposite of 9, which is 9." When viewing a standalone expression such as "$-k$," students might erroneously think this expression represents a negative number. However, if the value of k itself is a negative number, that is, if $k = -3$, then $-k = -(-3) = 3$. Thus, reading $-k$ as "the opposite of k" is a more accurate way of reading this expression. Teachers should be consistent in using the word "minus" only when referring to subtraction, and should use the word "negative" when referring to numbers like "-6" (that is, as opposed to saying "minus six").

470

471   In grade seven, students will explore operations with positive and negative rational
472   numbers, so it is important to develop a firm understanding of the relationship between
473   positive and negative numbers and their opposites here. Students recognize that a
474   number and its opposite are the same distance from 0 on a number line, as in 7.2 and
475   -7.2 being the same distance from 0:

476
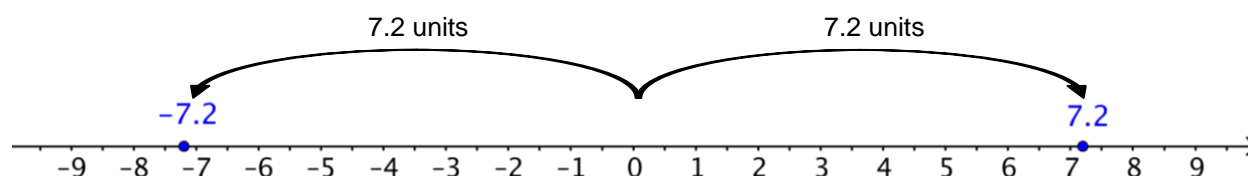
477   In addition, students recognize the minus sign as meaning "the opposite of," and that in
478   general the opposite of a number is the number on the other side of 0 at the same
479   distance from 0 as the original number, as in $-\left(-2\frac{1}{2}\right)$ is "the opposite of the opposite of
480   $2\frac{1}{2}$," which is just $2\frac{1}{2}$ again:

The opposite of $-2\frac{1}{2}$ is the number on the other side of 0, $2\frac{1}{2}$ units from 0.

481
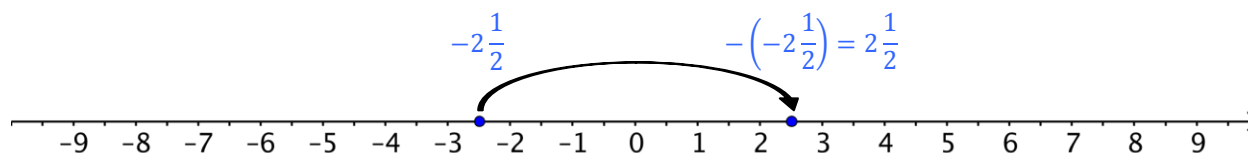
482   This understanding will help with later development of the notion of absolute value, as

483   the absolute value of a number is defined as its distance from 0 on a number line.

484

485   Students' previous work in the first quadrant grid helps them recognize the point where

486   the x-axis and y-axis intersect as the origin. Grade six students identify the four

487   quadrants and the appropriate quadrant for an ordered pair based on the signs of the

488   coordinates (**6.NS.6▲**). For example, students recognize that in Quadrant II, the signs

489   of all ordered pairs would be (−, +). Students understand the relationship between two

490   ordered pairs differing only by signs as reflections across one or both axes. For

491   example, in the ordered pairs (−2, 4) and (−2, −4), the y-coordinates differ only by

492   signs, which represents a reflection across the x-axis. A change in the x-coordinates

493   from (−2, 4) to (2, 4), represents a reflection across the y-axis. When the signs of both

494   coordinates change, for example, when (2, −4) changes to (−2, 4), the ordered pair is

495   reflected across both axes.

496

The Number System	6.NS

Apply and extend previous understandings of numbers to the system of rational numbers.

   7. Understand ordering and absolute value of rational numbers.

      a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.*

      b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write –3°C > –7°C to express the fact that –3°C is warmer than –7°C.*

      c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of –30 dollars, write |–30| = 30 to describe the size of the debt in dollars.*

      d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.*

   8. Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

497

498   In grade six students reason about the order and absolute value of rational numbers

499   (**6.NS.7▲**) and solve real world and mathematical problems by graphing in all four

500   quadrants of the coordinate plane (**6.NS.8▲**). Students use inequalities to express the

501   relationship between two rational numbers. Working with number line models helps

502   students internalize the order of the numbers—larger numbers on the right or top of the

503   number line and smaller numbers to the left or bottom of the number line. Students

504   correctly locate rational numbers on the number line, write inequalities, and explain the

505   relationships between numbers. Students understand the absolute value of a rational

506   number as its distance from 0 on the number line and recognize the symbols "| |" as

507   representing absolute value (e.g., $|3| = 3, |-2| = 2$). They distinguish comparisons of

508   absolute value from statements about order. (**6.NS.7**)

Example: Comparing Rational Numbers.

One of the thermometers shows $-3°C$ and the other shows $-7°C$. Which thermometer shows which temperature? Which is the colder temperature? How much colder? Write an inequality to show the relationship between the temperatures and explain how the model shows this relationship.

Solution: Since on a vertical number line, negative numbers get "more negative" as we go down the line, it appears that the thermometer on the left must read -7 and the thermometer on the right must read -3. By counting spaces they differ by, the thermometer on the left reads a temperature colder by 4 degrees. Related inequalities are $-7 < -3$ and $-3 > -7$.
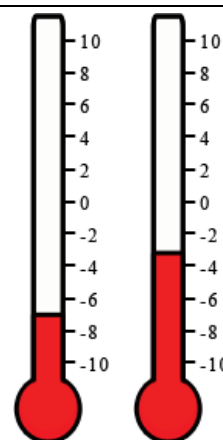
509

Common Misconceptions: With positive numbers the absolute value (distance from zero) of the number and the value of the number are the same. However students might be confused when they work with the absolute value of negative numbers. For negative numbers, as the value of the number decreases the absolute value increases. For example –24 is less than –14 because –24 is located to the left of –14 on the number line. However the absolute value of –24 is greater than the absolute value of –14. Students may also erroneously think that taking the absolute value means to "change the sign of a number" which is true for negative numbers but not for positive numbers or 0.

---

510   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

511

512   **Domain: Expressions and Equations**

513

514   A critical area of instruction at grade six is writing, interpreting and using expressions

515   and equations. In previous grades students wrote numerical equations and simple

516   equations involving one operation with a variable. In grade six students start the

517   systematic study of equations and inequalities and methods to solve them.

518

519   Students understand that mathematical expressions express calculations with numbers.

520   Some numbers might be given explicitly, like 2 or $\frac{3}{4}$. Other numbers are represented by

521   letters, such as $x, y, P,$ or n. The calculation an expression represents might use a single

522   operation, as in $4 + 3$ or $3x$, or a series of nested or parallel operations, as in $3(a + 9) -$

523   $\frac{9}{b}$. An expression can consist of a single number, even 0.

524

525   Students understand an equation is a statement that two expressions are equal. It is an

526   important aspect of equations that the two expressions on either side of the equal sign

527   might not actually always be equal; that is, the equation might be a true statement for

528   some values of the variable(s) and a false statement for others (Adapted from

529   Progressions 6-8 EE 2011.)

530

Expressions and Equations	6.EE

Apply and extend previous understandings of arithmetic to algebraic expressions.
1. Write and evaluate numerical expressions involving whole-number exponents.
2. Write, read, and evaluate expressions in which letters stand for numbers.
   a. Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as 5 – y.*
   b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.*
   c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas V = s^3 and A = 6 s^2 to find the volume and surface area of a cube with sides of length s = 1/2.*

531

532   Students demonstrate the meaning of exponents to write and evaluate numerical

533   expressions with whole number exponents. The base can be a whole number, positive

534   decimal, or a positive fraction (**6.EE.1▲**). Students should work with a variety of

535   expressions and problem situations to practice and deepen their skills. They can start

536   with simple expressions to evaluate and move to more complex expressions. For

537   example, they begin with simple whole numbers and move to fractions and decimal

538   numbers. (**MP.2, MP.6**)

539

Examples:

- What is the side length of a cube of volume 5^3 cubic cm? (5 cm)
- Write $10,000 = 10 \times 10 \times 10 \times 10$ with an exponent. (10^4)
- Andrea had half a pizza. She gave half of it to Marcus. Then Marcus gave half of what he had to Roger. Write the amount of pizza Roger has using exponents. $\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3 \right)$

Evaluate the Following:

- 4^3 (Answer: 4 × 4 × 4 = 64)
- $5 + 2^4 \cdot 6$ (Answer: $5 + 16 \cdot 6 = 5 + 96 = 101$)
- $7^2 - 24 \div 3 + 26$ (Answer: $49 - 8 + 26 = 67$)

540

541   This is a foundational year for building the bridge between concrete concepts of

542   arithmetic and the abstract thinking of algebra. Visual representations and concrete

543   models (such as algebra tiles, counters, and cubes) can help students translate

544   between concrete numerical representations and abstract symbolic representations.

545

Common Misconceptions: Students may not understand how to read the operations referenced with notations (e.g., x^3, $4x$, $3(x + 2y)$, $a + 3a$). Students are learning that

- x^3 means $x \cdot x \cdot x$, not $3x$ or 3 times x
- $4x$ means 4 times x or $x + x + x + x$, not forty-something
- When evaluating $4x$ when $x = 7$, substitution does not result in the expression meaning 47.
- For expressions like $a + 3a$, students need to see a as $1a$ to know that $a + 3a = 4a$ and not $3a^2$.

The use of the "x" notation as both the variable and the operation of multiplication can also be a source of confusion for students. In addition, students may need an explanation for why $x^0 = 1$ for all non-zero numbers x. Full explanations of this and other rules of working with exponents appear in grade eight.

---

546   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

547

548   Students write, read, and evaluate expressions in which letters stand for numbers.

549   (**6.EE.2▲**). Grade six students write expressions that record operations with numbers

550   and with letters standing for numbers. Students need opportunities to read algebraic

551   expressions to reinforce that the variable represents a number, and so behaves

552   according to the same rules for operations as numbers do (e.g. distributive property).

553

Examples of Interpreting Expressions
• The expression $r + 21$ represents "some (unknown) number plus 21."
• The expression $6 \cdot n$ represents "6 times some number n."
• The expression $\frac{s}{4}$ represents "s divided by 4", as well as "one-quarter of s."
• The expression $r - 4.5$ represents "the number r minus 4.5," or 4.5 less than r."
• The expression $3(x + 5)$ represents "3 times the sum of a number and 5."

554

555   Students identify the parts of an algebraic expression using mathematical terms such as

556   *variable*, *coefficient*, *constant*, *term*, *factor*, *sum*, *difference*, *product*, and *quotient*. They

557   should understand terms are the parts of a sum and when a term is an explicit number,

558   it is called a constant. When the term is a product of a number and a variable, the

559   number is called the coefficient of the variable. Variables are letters that represent

560   numbers. Development of this common language helps students understand the

561   structure of expressions and explain their process for simplifying expressions.

562

563   As students move from numerical to algebraic work, the multiplication and division

564   symbols $\times$ and $\div$ are replaced by the conventions of algebraic notation. Students learn

565   to use either a dot for multiplication, e.g., $1 \cdot 2 \cdot 3$ instead of $1 \times 2 \times 3$, or simple

566   juxtaposition, e.g., $3x$ instead of $3 \times x$, which is potentially confusing (during the

567   transition, students may indicate all multiplications with a dot, writing $3 \cdot x$ for $3x$).

568   Students also learn that $x \div 2$ can be written as $\frac{x}{2}$. (Adapted from Progressions 6-8 EE

569   2011.)

570

Examples of Expression Language. In the expression $x^2 + 5y + 3x + 6$,
• The variables are x and y.

- There are 4 terms, x^2, $5y$, $3x$, and 6.
- There are 3 variable terms, x^2, $5y$, $3x$. They have coefficients of 1, 5, and 3 respectively.
- The coefficient of x^2 is 1, since $x^2 = 1 \cdot x^2$.
- The term $5y$ represents $y + y + y + y + y$ or $5 \cdot y$.
- There is one constant term, 6.
- The expression shows a sum of all four terms.

571

572 Sixth grade students evaluate various expressions at specific values of their variables,

573 including expressions that arise from formulas used in real-world problems. Examples

574 where students evaluate the same expression at several different values of a variable

575 are important for the later development of the concept of a *function,* and these should

576 be experienced more frequently than problems wherein the values of the variables stay

577 the same and the expression continues to change. **(MP.1, MP.2, MP.3, MP.4, MP.6)**

578

Examples of Evaluating Expressions.
1. Evaluate the two expressions $5(n + 3) + 7n$ and $12n + 15$ for $n = -2, 0, \frac{1}{2}, 7.5$. What do you notice?
2. The expression $c + 0.07c$ can be used to find the total cost of an item with 7% sales tax added, where c is the cost of the item before taxes are added. Use this expression to find the total cost of an item that costs $25. Now for an item that costs $250. Now for an item that costs $25,000.
3. The perimeter of a parallelogram is found using the formula $P = 2l + 2w$. What is the perimeter of a rectangular picture frame with dimensions of 8.5 inches by 11 inches?

579 (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

580

Expressions and Equations	6.EE
Apply and extend previous understandings of arithmetic to algebraic expressions.	

3. Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*
4. Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

581

582   Students use their understanding of multiplication to interpret $3(2 + x)$ as 3 groups of

583    $(2 + x)$ or the area of a rectangle of lengths 3 units and $(x + 2)$ units. (**MP.2, MP.3,**

584   **MP.4, MP.6, MP.7**) They use a model to represent x and make an array to show the

585   meaning of $3(2 + x)$. They can explain why it makes sense that $3(2 + x)$ is equal to

586    $6 + 3x$. Manipulatives such as "algebra tiles," which make use of the area model to

587   represent quantities, can be used to show why this is true. Note that with algebra tiles a

588   1×1 square represents a unit (the number 1), while the variable x is represented by a

589   rectangle of dimensions $1 \times x$ (that is, the longer side of the x-tile is not commensurate

590   with a whole number of unit tiles, and therefore represents an "unknown" length).

Example of Basic Reasoning with Algebra Tiles:

Students can recognize $3(x + 2)$ as representing the area of a rectangle of lengths 3 units and $(x + 2)$ units. Using the appropriate number of tiles (or a sketch), students can see that there are $3 \cdot 2 = 6$ and $3 \cdot x = 3x$ units altogether, so that $3(x + 2) = 3x + 6$.

591
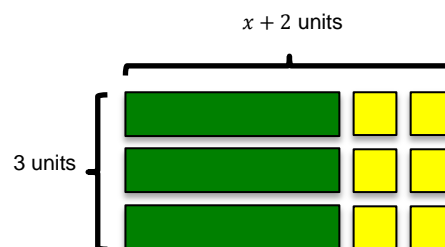
592   Important in standards (**6.EE.3▲**) and (**6.EE.4▲**) is for students to understand that the

593   distributive property is the basis for combining "like" terms in an expression (or

594   equation). For instance, students understand that $4a + 7a = 11a$, because

$$4a + 7a = (4 + 7)a = 11a.$$

595   This ability to use the distributive property "forwards" and "backwards" is important for

596   students to develop. Students generate equivalent expressions in general using the

597   associative, commutative, and distributive properties and can prove the expressions are

598   equivalent (**MP.1, MP.2, MP.3, MP.4, MP.6).**

599

Example: Equivalent Expressions.

Show that the two expressions $5(n + 3) + 7n$ and $12n + 15$ are equivalent.

Solution: By applying the distributive property, I know that $5(n + 3) + 7n$ can be rewritten as $5n + 15 + 7n$ because of the distributive property. Also, since $5n + 7n = (5 + 7)n = 12n$, I can write the expression as $12n + 15$.

---

600   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

601

Expressions and Equations	6.EE

Reason about and solve one-variable equations and inequalities.

5. Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
6. Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
7. Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.
8. Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

602

603   In the elementary grades students explored the concept of equality. In sixth grade

604   students explore equations as one expression being set equal to a specific value. A

605   solution is a value of the variable that makes the equation true. Students use various

606   processes to identify such value(s) that when substituted for the variable will make the

607   equation true (**6.EE.5▲**). Students can use manipulatives and pictures (e.g., tape-like

608   diagrams) to represent equations and their solution strategies. When writing equations,

609   students learn to be precise in their definition of a variable, e.g., writing "n equals John's

610   age in years" as opposed to simply writing "n is John." (**6.EE.6▲**). (MP.6)

611

Examples: Solving Equations of the Form $p + x = q$ and $px = q$. (6.EE.7▲).

1. Joey had 26 game cards. His friend Richard gave him some more and now he has 100 cards. How many cards did his friend Richard give him? Write and equation and solve your equation.

    Solution: Since Richard gave him some more cards, we let n be the number of cards that Richard gave Joey. This means he now has $26 + n$ cards. But the number of cards Joey has is 100, so we get the equation $26 + n = 100$. Using the relationship between addition and subtraction, we see that $n = 100 - 26 = 74$, which means that his friend gave him 74 cards.   One can represent this equation with a tape-like diagram:

100	
26	n

2. A book of Theme Park Ride tickets costs $30.00. Each ticket on its own costs $2.50. How many tickets come in each book? Write and solve an equation that represents this situation.

The *Mathematics Framework* was adopted by the California State Board of Education on November 6, 2013. *The Mathematics Framework* has not been edited for publication.

Solution: If s represents the number of stamps in one booklet, then $(2.50)s$ is the cost of s tickets in dollars. Since the cost of one book is $30.00, solving the equation $(2.50)s = 30.00$ would give the number of tickets. To solve this, we realize that if $2.50 \times s = 30.00$, then $s = 30.00 \div 2.50 = 12$. This means there are 12 tickets in each book.

3. Meagan spent $56.58 on three pairs of jeans. If each pair of jeans costs the same amount, write an equation that represents this situation and solve it to determine how much one pair of jeans cost.

$56.58		
J	J	J

Solution: If J represents the cost of one pair of jeans in dollars, then the equation becomes $3J = 56.58$. If we solve this for J, we find $J = 56.58 \div 3 = 18.86$. This means each pair of jeans cost $18.86.

4. Julio gets paid $20 for babysitting. He spent $1.99 on a package of trading cards and $6.50 on lunch. Write and solve an equation to show how much money Julio has left.

Solution: One equation might be $1.99 + 6.50 + x = 20.00$, where x represents how many dollars he has left. We find that $x = 11.51$, so that he has $11.51 left.

612 (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

613

614 Many real-world situations are represented by inequalities. In grade six, students write
615 simple inequalities involving $<$ or $>$ to represent real world and mathematical situations,
616 and they use the number line to model the solutions (**6.EE.8▲**). Students learn that
617 when representing inequalities of these forms on a number line, the common practice is
618 to draw an arrow on or above the number line with an open circle on or above the
619 number in the inequality. The arrow indicates the numbers greater than or less than the
620 number in question, and that the solutions extend indefinitely. The arrow is a solid line,
621 to indicate that even fractional and decimal amounts (i.e. points between dashes on the
622 line) are included in the solution set.

623

Examples: Inequalities of the Form $x < c$ and $x > c$.

1. The class must raise more than $100 to go on the field trip. Let m represent the amount of money in dollars that the class raises. Write an inequality that describes how much the class needs to raise. Represent this on a number line.

Solution: Since the amount of money, m, needs to be greater than 100, the inequality is $m > 100$. A number line diagram for this might look like:

2. The Flores family spent less than $50.00 on groceries last week. Write an inequality that describes this situation and graph the solution on a number line.

    Solution: If we let g represent the amount of money in dollars the family spent on groceries last week, then the inequality becomes $k < 50$. We might represent this in the following way:
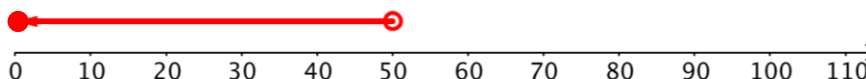
    (In this example, it doesn't make sense that the Flores family could have spent a negative amount of dollars on groceries, so the arrow would stop precisely at $0; we would typically represent this with a dot over 0 rather than the arrow.)

3. Graph $x < 4$.

    Solution: This represents all numbers less than 4:
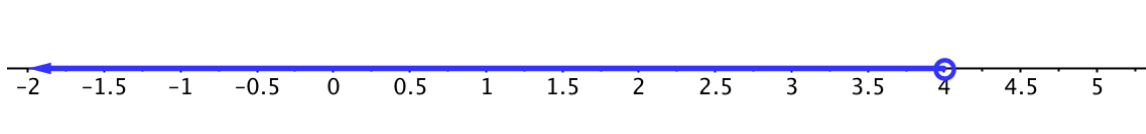
624   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

625

Expressions and Equations	6.EE

Represent and analyze quantitative relationships between dependent and independent variables.
  9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.*

626

627   In grade six students investigate the relationship between two variables, beginning with

628   the distinction between dependent and independent variables (**6.EE.9▲**). The

629   independent variable is the variable that can be changed; the dependent variable is the

630   variable that is affected by the change in the independent variable. Students recognize

631   that the independent variable is graphed on the x-axis; the dependent variable is

632   graphed on the y-axis. They also understand that not all data should be graphed with a

633   line. Data that is discrete would be graphed with coordinates only.

634

635   Students represent relationships between quantities with multiple representations,

636   including describing the relationship using language, a table, an equation, or a graph.

637   Translating between multiple representations helps students understand that each form

638   represents the same relationship and provides a different perspective on the

639   relationship.

640

Example: Exploring Dependent and Independent Variables. (Adapted from illustrativemathematics.org, 6.EE Chocolate Bar Sales.)

Stephanie is helping her band collect money to fund a filed trip. The band decided to sell boxes of chocolate bars. Each bar sells for $1.50 and contains 20 bars. Shown is a partial table of money collected for different numbers of boxes sold.

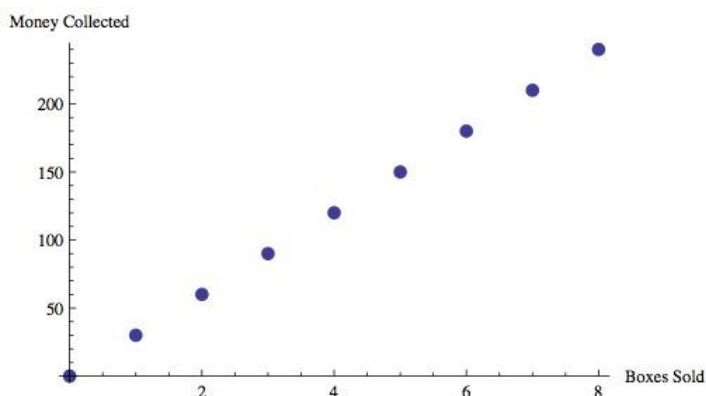| Boxes Sold b	Money Collected m
1	$30.00
2	
3	
4	
5	$150.00
6	
7	
8	

a.   Complete the table for the remaining values of m.

b.   Write an equation for the amount of money, m, that will be collected if b boxes of bars are sold. Which is the independent variable and which is the dependent variable?

c.   Graph the relationship using ordered pairs found from the table.

d.   Calculate how much money will be collected if 100 boxes of chocolate bars are sold.

Solutions:

b.   Students may derive the equation $m = 30b$, representing the fact that when b boxes are sold at $30 per box, then the total amount of money collected is $30b$ dollars. In this case, the independent variable is the number of boxes sold, b, while the money collected is the dependent variable. This certainly is a valid way to make sense of the problem, in that the amount of money collected depends on the number of boxes sold.

However, if one has fundraising goals, then it would be natural to think of the relationship as $b = \frac{m}{30}$, in the sense that the number of boxes needed to be sold depends on the fundraising target.

c.   If we graph the relationship as (b, m), then we obtain the graph

The *Mathematics Framework* was adopted by the California State Board of Education on November 6, 2013. *The Mathematics Framework* has not been edited for publication.

> shown, which illustrates the relationship $m = 30b$. (In grade seven, students will more fully explore
>
> graphs of proportional relationships such as this one.)

641

Domain: Geometry

643

644   In grade six students extend their understanding of length, area, and volume as they

645   solve problems by applying formulas for the area of triangles and parallelograms and

646   volume of rectangular prisms.

647

Geometry	6.G

Solve real-world and mathematical problems involving area, surface area, and volume.
1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\,w\,h$ and $V = b\,h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

648

649   Students in grade six build on their work with area in earlier grades by reasoning about

650   relationships among shapes to determine area, surface area, and volume. Students

651   continue to understand area as the number of squares needed to cover a plane figure.

652   Sixth grade students find areas of right triangles, other triangles, and special

653   quadrilaterals by decomposing these shapes, rearranging or removing pieces, and

654   relating the shapes to rectangles. As students compose and decompose shapes to

655   determine areas, they learn that area is conserved when composing or decomposing

656   shapes. For example, students will decompose trapezoids into triangles and/or

657   rectangles, and use this reasoning to find formulas for the area of a trapezoid. Students

658   know area formulas for triangles and some special quadrilaterals, in the sense of having
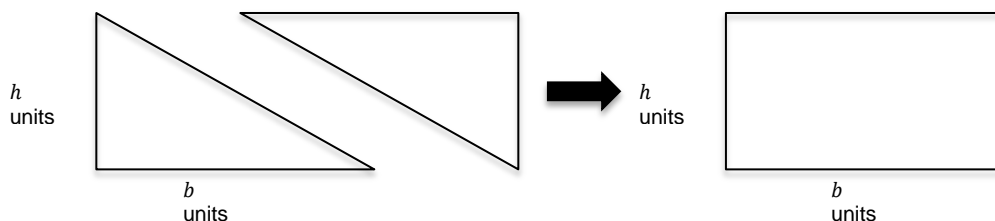
659   an understanding of why the formula works and how the formula relates to the measure

660   (area) and the figure **(6.G.1).**

661

662   Prior to seeing formulas for areas of different shapes, students can find areas of shapes

663   on centimeter grid paper by duplicating, composing, and decomposing shapes. These

664   experiences will make them familiar with the processes that result in the derivations of
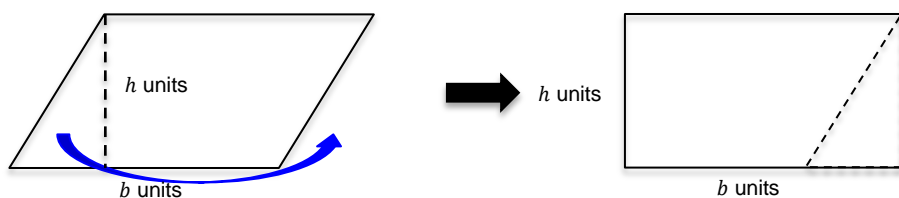
665   the formulas shown below.

666

Deriving Area Formulas (MP.3, MP.7).
Starting with a basic understanding of the area of a rectangle of base b units and height h units being bh square units, along with the relationship between rectangles and triangles, and the law of conservation of area, students can justify area formulas for various shapes.
Right Triangles: Since two right triangles of base b and height h can be composed to form a rectangle of the same base and height, the triangle must have an area ½ that of the rectangle. Thus, the area of a right triangle of base b and height h is $\frac{1}{2}bh$ square units.
Parallelograms: If we define the height of the parallelogram to be the length of a perpendicular segment from base to base, then a parallelogram of base b and height h has the same area (bh square units) as a rectangle of the same dimensions. We cut off a right triangle as shown, and move it to complete the rectangle.
Non-Right Triangles: Non-right triangles of base b units and height h units can now be duplicated to make parallelograms. By similar reasoning used with right triangles and rectangles, the area of such a triangle is $\frac{1}{2}bh$ square units. (One can show the same holds true for obtuse triangles.)

**Trapezoids:** Trapezoids can be deconstructed into two triangles of bases a and b, showing that the area of a trapezoid can be found by $\frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}(a+b)h$.
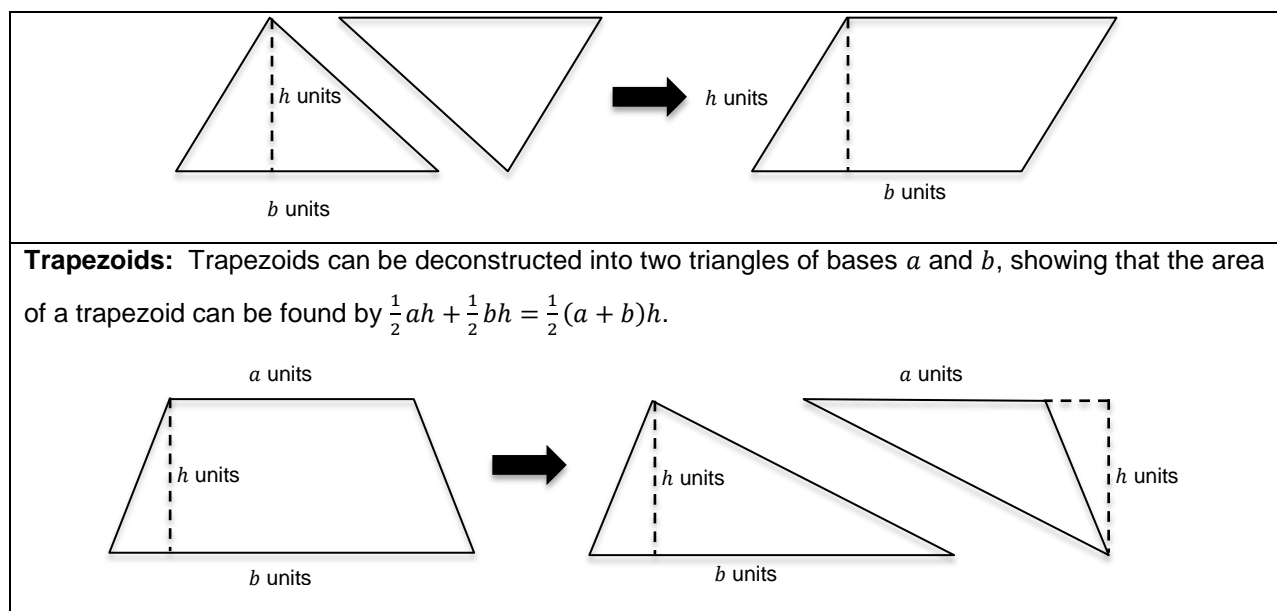
667

668   Previously, students calculated the volume of right rectangular prisms using whole

669   number edges, and understood doing so as finding the number of unit cubes (i.e.

670   $1 \times 1 \times 1$ cubic unit) within a solid shape. In grade six students extend this work to unit

671   cubes with fractional edge lengths, e.g., they find volumes by finding the number of

672   units cubes of dimensions $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ within a figure with fractional side lengths. Students

673   draw diagrams to represent fractional side lengths, connecting finding these volumes

674   with multiplication of fractions. **(6.G.2)**

675

Example: Counting Fractional Cubic Units.

The model shows a rectangular prism with dimensions $\frac{3}{2}$ inches, $\frac{5}{2}$ inches, and $\frac{5}{2}$ inches. Each of the cubic units shown in the model has a volume of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ cubic inches. Students should reason why each of these units has this volume (i.e. by discovering that 8 of them fit in a $1 \times 1 \times 1$-cube). Furthermore, students explain why the volume of the rectangular prism is given by $\frac{3}{2} \times \frac{5}{2} \times \frac{5}{2}$ cubic inches, and why one can also find the volume by finding $(3 \times 5 \times 5) \times \left(\frac{1}{8} \text{ cubic inch}\right)$.

$\frac{3}{2}$ in.

$\frac{5}{2}$ in.

$\frac{5}{2}$ in.
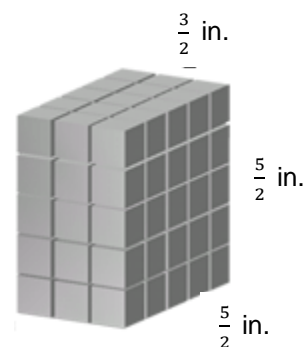
676   (Adapted from Arizona 2012)

677

678   Finding areas, surface areas and volumes present modeling opportunities (**MP.4**) and

679   require students to attend to precision with the types of units involved (**MP.6**).

680

681   In standard **6.G.3**, students represent shapes in the coordinate plane. They find lengths

682   of sides that contain vertices with a common x- or y-coordinate, representing an

683   important step for later grade eight understanding of how to use the distance formula to

684   find the distance between any two points in the plane. In addition, in grade six students

685   construct three-dimensional shapes using nets and build on their work with areas

686   (**6.G.4**) by finding surface areas using nets.

687

Example: Polygons in the Coordinate Plane.
On a grid map, the library is located at (−2, 2), the city hall building is located at (0, 2), and the high school is located at (0, 0). a.   Represent the locations as points on a coordinate grid with a unit of 1 mile. b.   What is the distance from the library to the city hall building? c.   What is the distance from the city hall building to the high school? How do you know? d.   What is the shape that results from connecting the three locations with line segments? e.   The city council is planning to place a city park in this area. What is the area of the planned park?

688
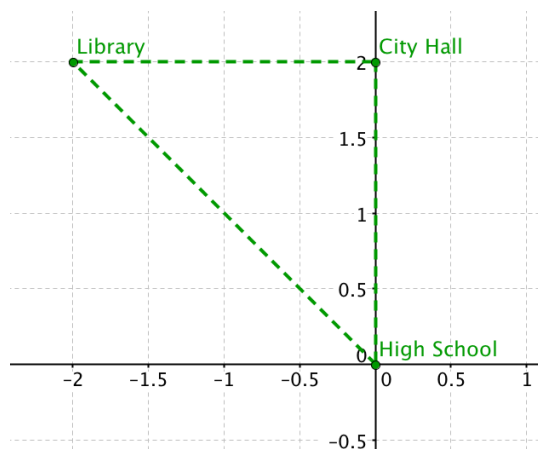
689   [Note: Sidebar]

Focus, Coherence, and Rigor.
The standards in the cluster "Solve real-world and mathematical problems involving area, surface area, and volume" regarding areas of triangles and volumes of right rectangular prisms (**6.G.1 -2**) connect to major work in the Expressions and Equations domain (**6.EE.1-9▲**). In addition, standard **6.G.3** asks students to draw polygons in the coordinate plane, which supports major work with the coordinate plane in the Number System domain (**6.NS.8▲**).

690

691   **Domain: Statistics and Probability**

692   A critical area of instruction in grade six is developing understanding of statistical

693   thinking. Students build on their knowledge and experiences in data analysis as they

694   work with statistical variability and represent and analyze data distributions. They

695   continue to think statistically, viewing statistical reasoning as a four-step investigative

696   process:

Four-Step Statistical Investigation.
• Formulate questions that can be answered with data.
• Design and use a plan to collect relevant data.
• Analyze the data with appropriate methods.
• Interpret results and draw valid conclusions from the data that relate to the questions posed.

697   (Progressions, 6-8 SP 2011)

698

Statistics and Probability	6.SP
Develop understanding of statistical variability. 1. Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.* 2. Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape. 3. Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.	

699

700   Statistical investigations start with questions, which can result in a narrow or wide range

701   of numerical values, and ultimately result in some degree of variability **(6.SP.1)**. For

702   example, asking classmates "How old are the students in my class in years?" will result

703   in less variability than asking "How old are the students in my class in months?"

704   Students understand questions need to specifically demand measureable answers, e.g.,

705   if a student wants to know about the exercise habits of the students at their school,

706   rather than asking "Do you exercise?" a statistical question for this study could be "How

707   many hours per week on average do students at Jefferson Middle School exercise?"

708   Grade six students design survey questions that anticipate variability. They understand

709   surveys include a variety of possible responses with specific numerical answers (e.g., 3

710   hours per week, 4 hours per week).

711   (Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

712

713   A major focus of grade six is the characterization of data distributions by measures of

714   center and spread. To be useful, center and spread must have well-defined numerical

715   descriptions that are commonly understood by those using the results of a statistical

716   investigation (Progressions 6-8 SP 2011). Sixth grade students analyze the center,

717   spread, and overall shape of a set of data **(6.SP.1)**. As students analyze and/or

718   compare data sets they consider the context in which the data is collected and identify

719   clusters, peaks, gaps, and symmetry in the data. Students learn that data sets contain

720   many numerical values that can be summarized by one number such as a measure of

721   center (mean and median) and range.

722

Describing Data.

The *measure of center* gives a numerical value to represent the center of the data (e.g., midpoint of an ordered list or the balancing point). The *range* provides a single number that describes how the values vary across the data set. Another characteristic of a data set is the measure of *variability* (or spread) of the values.

723

724   **Measures of Center**

725   Given a set of data values, students summarize the measure of center with the median

726   or mean **(6.SP.3)**. The *median* is the value in the middle of an ordered list of data. This

727   value means that 50% of the data is greater than or equal to it and that 50% of the data

728   is less than or equal to it. When there is an even number of data values, the median is

729   the arithmetic average of the two in the middle.

730

731   The *mean* is the arithmetic average: the sum of the values in a data set divided by the

732   number of data values in the set. The mean measures center in the sense that it is the

733   hypothetical value that each data point would equal if the total of the data values were

734   redistributed equally. Students can develop an understanding of what the mean

735   represents by redistributing data sets to be level or fair (creating an equal distribution)

736   and by observing that the total distance of the data values above the mean is equal to

737   the total distance of the data values below the mean (reflecting the idea of a balance

738   point).

739

Example: Representing Data and Finding Measures of Center.

Consider the data shown in the following dot plot of the scores for organization skills for a group of students.

6-Trait Writing Rubric
Scores for Organization
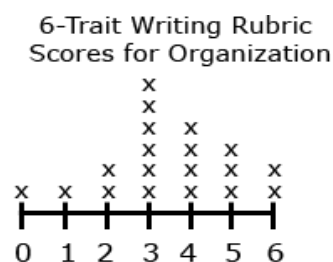
a.   How many students are represented in the data set?

b.   What are the mean and median of the data set? Compare the mean and median.

c.   What is the range of the data? What does this value tell you?

Solution:

a.   Since there are 19 data points (represented by X's) in the set, there are 19 students represented.

b.   The mean of the data set can be found by adding all of the data values (scores) and dividing by 19 (the calculation below is recorded as [score] × [number of students with that score])

$$\frac{0(1) + 1(1) + 2(2) + 3(6) + 4(4) + 5(3) + 6(2)}{19} = \frac{66}{19} \approx 3.47$$

The median of the data set appears to be 3. To check this, we can line up the data values and look for the center:

$$0, 1, 2, 2, 3, 3, 3, 3, 3, \mathbf{\color{red}3}, 4, 4, 4, 4, 5, 5, 5, 6, 6$$

The median is indeed 3, since there are 9 data values to the left and 9 values to the right of 3. The mean is greater than the median, which makes sense because the data is slightly skewed to the right.

c.   The range of the data is 6, which happens to coincide with the range of scores possible.

740

741   **Measures of Variability**

742   In grade six, variability is measured using the interquartile range or the mean absolute

743   deviation. The *interquartile range* (IQR) describes the variability between the middle

744   50% of a data set. It is found by subtracting the lower quartile from the upper quartile. In

745   a box plot, it represents the length of the box and is not affected by outliers. Students

746   find the IQR from a data set by finding the upper and lower quartiles and taking the

747   difference, or from reading a box plot.

748

749   *Mean absolute deviation* (MAD) describes the variability of the data set by determining

750   the absolute deviation (the distance) of each data piece from the mean, and then finding

751   the average of these deviations. Both the interquartile range and the Mean Absolute

752   Deviation are represented by a single numerical value. Higher values represent a

753   greater variability in the data.

754

Example: Finding the IQR and MAD
In the example above, the data set was $0, 1, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6$. The median (3) separated the data set into an upper and lower 50%. By further separating these two subsets, we obtain the four *quartiles* (i.e., 25%-sized parts of the data set). $$0, \ 1, \ 2, \ 2, \ 3, \ 3, \ 3, \ 3, \ 3, \ 3, \ 4, \ 4, \ 4, \ 4, \ 5, \ 5, \ 5, \ 6, \ 6$$ In this case, the IQR is $5 - 3 = 2$, indicating that the middle 50% of values differ by no more than 2 units. This is reflected in the dot plot, as most of the data appears to be clustered around 3 and 4.
To find the MAD of the data set above, we'll round the mean to 3.5 to simply our calculations, and find that there are 6 possible deviations from the mean: $$|0 - 3.5|, |1 - 3.5|, |2 - 3.5|, |3 - 3.5|, |4 - 3.5|, |5 - 3.5|, |6 - 3.5|,$$ resulting in the set of deviations $3.5, 2.5, .5, .5, 1.5, 2.5$. When we find the average of these deviations, we obtain: $$\frac{1(3.5) + 1(2.5) + 2(1.5) + 6(.5) + 4(.5) + 3(1.5) + 2(2.5)}{19} \approx 1.24.$$ This is interpreted as saying that on average, a student's score was 1.24 points away from the approximate mean of 3.5.
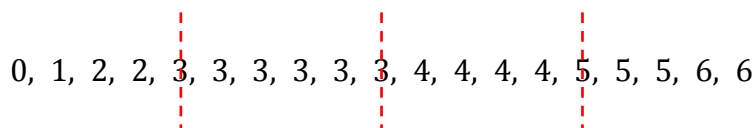
755

756

<table>
<tr><td colspan="2">Statistics and Probability   6.SP</td></tr>
<tr><td colspan="2">

Summarize and describe distributions.
4. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
5. Summarize numerical data sets in relation to their context, such as by:
   a. Reporting the number of observations.
   b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.
   c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.
   d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

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757

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758   Grade six students display data graphically using number lines, as well as dot plots,

759   histograms, and box plot graphs **(6.SP.4)**. Students learn to determine the appropriate

760   graph to use to display data and how to read data from graphs generated by others.

761

Graphical Displays of Data in Grade Six.

- *Dot plots* are simple plots on a number line where each dot represents a piece of data in the data set. Dot plots are suitable for small to moderate size data sets and are useful for highlighting the distribution of the data including clusters, gaps, and outliers.

- A *histogram* shows the distribution of data using intervals on the number line. The height of each bar represents the number of data values in that interval. In most real data sets, there is a large amount of data and many numbers will be unique.

- A *box plot* shows the distribution of values in a data set by dividing the set into quartiles. It can be graphed either vertically or horizontally. The box plot is constructed from the five-number summary (*minimum, lower quartile, median, upper quartile,* and *maximum*). These values give a summary of the shape of a distribution. Box plots display the degree of spread of the data and the skewness of the data and can help students compare two sets of data.

762

763   Students in grade six interpret data displays and determine measures of center and

764   variability from them. They summarize numerical data sets in relation to their context

765   **(6.SP.5).**

766

Examples: Interpreting Data Displays.

1. Grade six students were collecting data for a math class project. They decided they would survey the other two grade six classes to determine how many video games each student owns. A total of 38 students were surveyed. The data are shown in the table below in no specific order. Create a data display. What are some observations that can be made from the data display?

| 11	21	5	12	10	31	19	13	23	33
10	11	25	14	34	15	14	29	8	5
22	26	23	12	27	4	25	15	7	
2	19	12	39	17	16	15	28	16	

Solution: Students might make a histogram with 4 ranges (0-9,10-19, 20-29, 30-29) to display the data. It appears from the histogram that the mean and median are somewhere between 10-19, since the data of so many students lies in this range. Relatively few students own more than 30 video games, in fact, further analysis may prove the data point 39 to be an outlier.

---

2. Ms. Wheeler asked each student in her class to write their age in months on a sticky note. The 28 students in the class brought their sticky note to the front of the room and posted them in order on the white board. The data set is listed below in order from least to greatest. Create a data display. What are some observations that can be made from the data display?

Students Own
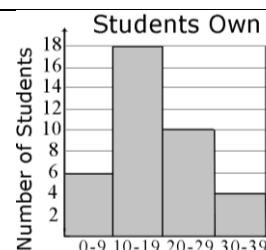
| 130 | 130 | 131 | 131 | 132 | 132 | 132 | 133 | 134 | 136 |
| 137 | 137 | 138 | 139 | 139 | 139 | 140 | 141 | 142 | 142 |

Solution: By finding the **five number summary** of the data, we can create a box plot. The minimum data value is 130 months, the maxium 150 months, and the median 139 months. To find the first quartile (Q_1) and second quartile (Q_3), we find the middle of the upper and lower 50%. Since there is an even number of data points in each of these parts, we must find the average, so that $Q_1 = \frac{132+133}{2} = 132.5$ and $Q_3 = \frac{142+143}{2} = 142.5$. Thus, the five number summary is:

Ages in Months of a Class of 6th Grade Students
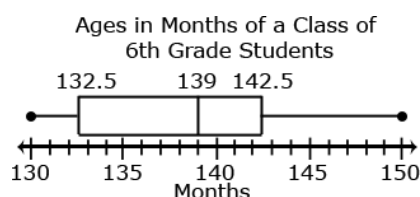
Minimum	First Quartile (Q_1)	Median (Q_2)	Third Quartile (Q_3)	Maximum
130	132.5	139	142.5	150

Now, a box plot is easy to construct. The box plot helps us see that the middle 50% of values lie between 132.5 and 142.5 months. Meanwhile, only 25% of values are between 130 and 132.5 months, and only 25% of values are between 142.5 and 150.

767   Adapted from Arizona 2012, N. Carolina 2012, and KATM 6th FlipBook 2012)

768

Focus, Coherence, and Rigor:

As students display and summarize numerical data (**6.SP.4-5**) they also support mathematical practices such as make sense of given data (**MP.1**), appropriate models and uses of statistical data and measures (**MP.4**, **MP.5**), and precision in finding and applying statistical measures (**MP.6**).

769

770   Students can use applets to create data displays. For example:

771   • Box Plot Tool http://illuminations.nctm.org/ActivityDetail.aspx?ID=77

772   • Histogram Tool http://illuminations.nctm.org/ActivityDetail.aspx?ID=78

773   (NCTM Illuminations 2013)

774

775   **Essential Learning for the Next Grade**

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776 In middle school, multiplication and division develop into powerful forms of ratio and

777 proportional reasoning. The properties of operations take on prominence as arithmetic

778 matures into algebra. The theme of quantitative relationships also becomes explicit in

779 grades six through eight, developing into the formal notion of a function by grade eight.

780 Meanwhile, the foundations of high school deductive geometry are laid in the middle

781 grades. The gradual development of data representations in kindergarten through grade

782 five leads to statistics in middle school: the study of shape, center, and spread of data

783 distributions; possible associations between two variables; and the use of sampling in

784 making statistical decisions (Adapted from the Partnership for Assessment of

785 Readiness for College and Careers [PARCC] 2012).

786

787 To be prepared for grade seven mathematics students should be able to demonstrate

788 they have acquired certain mathematical concepts and procedural skills by the end of

789 grade six and have met the fluency expectations for the grade six. For sixth graders, the

790 expected fluencies are multi-digit whole number division (**6.NS.2**) and multi-digit decimal

791 operations (**6.NS.3**). These fluencies and the conceptual understandings that support

792 them are foundational for work with fractions and decimals in grade seven.

793

794 Of particular importance for students to attain in grade six are skills and understandings

795 of division of fractions by fractions (**6.NS.1▲**); an understanding of the system of

796 rational numbers (**6.NS.5-8▲**); the ability to use ratio concepts and reasoning to solve

797 problems (**6.RP.1-3▲**); the extension of arithmetic to algebraic expressions (**6.EE.1-**

798 **4▲)**, including how to reason about and solve one-variable equations and inequalities

799 (**6.EE.8-4▲)**; and the ability to represent and analyze quantitative relationships between

800 dependent and independent variables (**6.EE.1-9▲)**.

801

802 Guidance on Course Placement and Sequences

803 The Common Core standards for grades six through eight are comprehensive, rigorous,

804 and non-redundant. Acceleration will require compaction not the former strategy of

805 deletion. Therefore, careful consideration needs to be made before placing a student

806 into higher mathematics coursework in middle grades. Acceleration may get students to

807   advanced coursework but might create gaps in students' mathematical background.

808   Careful consideration and systematic collection of multiple measures of individual

809   student performance on both the content and practice standards will be required. For

810   additional information and guidance on course placement, see Appendix A: Course

811   Placement and Sequences in this framework.

812

813

814 ## Grade 6 Overview
815

816 ### Ratios and Proportional Relationships

817 • Understand ratio concepts and use ratio reasoning to solve
818 problems.

819

820 ### The Number System

821 • Apply and extend previous understandings of multiplication and
822 division to divide fractions by fractions.

823 • Compute fluently with multi-digit numbers and find common
824 factors and multiples.

825 • Apply and extend previous understandings of numbers to the
826 system of rational numbers.

827

828 ### Expressions and Equations

829 • Apply and extend previous understandings of arithmetic to
830 algebraic expressions.

831 • Reason about and solve one-variable equations and inequalities.

832 • Represent and analyze quantitative relationships between dependent and independent
833 variables.

834

835 ### Geometry

836 • Solve real-world and mathematical problems involving area, surface area, and volume.

837

838 ### Statistics and Probability

839 • Develop understanding of statistical variability.

840 • Summarize and describe distributions.

841

Mathematical Practices

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.

5. Use appropriate tools strategically.

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

---

842   # Grade 6

Ratios and Proportional Relationships	6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*

2. Understand the concept of a unit rate *a/b* associated with a ratio *a:b* with *b* ≠ 0, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*[1]

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

    a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

    b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*

    c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

    d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

The Number System	6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?*

Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.

3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*

843
844
845
846 _____
847 [1]Expectations for unit rates in this grade are limited to non-complex fractions.

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Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

   a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

   b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

   c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

7. Understand ordering and absolute value of rational numbers.

   a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret $-3 > -7$ as a statement that -3 is located to the right of -7 on a number line oriented from left to right.*

   b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write $-3°C > -7°C$ to express the fact that $-3°C$ is warmer than $-7°C$.*

   c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of -30 dollars, write $|-30| = 30$ to describe the size of the debt in dollars.*

   d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.*

8. Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Expressions and Equations   6.EE

Apply and extend previous understandings of arithmetic to algebraic expressions.

1. Write and evaluate numerical expressions involving whole-number exponents.

2. Write, read, and evaluate expressions in which letters stand for numbers.

   a. Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as $5 - y$.*

   b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression $2(8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.*

   c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$.*

3. Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*

848

4. Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

Reason about and solve one-variable equations and inequalities.

5. Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
6. Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
7. Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.
8. Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Represent and analyze quantitative relationships between dependent and independent variables.

9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.*

Geometry   6.G

Solve real-world and mathematical problems involving area, surface area, and volume.

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\,w\,h$ and $V = b\,h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics and Probability   6.SP

Develop understanding of statistical variability.

1. Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
2. Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.
3. Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Summarize and describe distributions.

4. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

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5. Summarize numerical data sets in relation to their context, such as by:
    a. Reporting the number of observations.
    b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.
    c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.
    d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

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